NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XIX | Baton Rouge, La., October, 1944 | No. 1

[illegible] PER YEAR : SINGLE COPIES 40c.

Math. Lib.
Faxon

SUBSCRIPTION
$3.00 PER YEAR
IN ADVANCE
SINGLE COPIES
40c.

ALL BUSINESS
SHOULD BE ADDRESSED
TO THE
EDITOR AND MANAGER

VOL. XIX | BATON ROUGE, LA., October, 1944 | No. 1

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

Bright Skies!

Throughout the summer new and renewing subscribers to the Magazine kept the Baton Rouge office busy crediting pre-October subscription checks in accordance with our April and May announcements. The picture was complete when in August a gift of $200.00 came from the Mathematical Association of America.

All honor to the Mathematical Association of America!

All honor to the hundreds of loyal Magazine supporters whose faith in this journal is an unfailing inspiration!

We know we speak for our editorial colleagues in this declaration of renewed effort to present to its readers material of maximum quality in every division of National Mathematics Magazine.

S. T. Sanders.

Tensor Algebra and Invariants, I

By T. L. Wade
Florida State College for Women

1. *Introduction.* This paper has two purposes: to give an introduction to tensor algebra through the more familiar matrix algebra; and to indicate how tensor algebra may be used advantageously in the study of algebraic invariants under general and restricted linear transformations. Some indication of a renewal of interest in invariant theory is reflected in the recent authoritative book by H. Weyl[1]; as stated in the preface of that book one of its aims is to give a modern introduction to the theory of invariants; further, on page 28, therein, it is stated, "In recent times the tree of invariant theory has shown new life, and has begun to blossom again, chiefly as a consequence of the invariant-theoretic questions awakened by the revolutionary developments in mathematical physics (relativity theory and quantum mechanics), but also due to the connection of invariant theory with the extension of the theory of representations to continuous groups and algebras." The relation of invariants to elementary mathematics has lately been discussed in a stimulating way in this Magazine by H. V. Craig.[2]

Familiarity with the basic concepts and elementary operations of matrix algebra is assumed. Matrices used here are real; furthermore, they are square, or they have one row or one column. A square matrix with n rows and n columns will be represented by $[a]^n_n$; the transpose of this matrix, that is, the matrix obtained from $[a]^n_n$ by interchanging rows and columns, we denote by $\overline{\underline{a}}^{\,n}_{\,n}$. A one-rowed matrix with n elements we represent by $[a]_n$, that is, $[a]_n=[a_1, a_2,\cdots, a_n]$; and its transpose, a one-column matrix, we represent by $\overline{\underline{a}}^{\,n}$. The unit matrix will be denoted by $[1]^n_n$. The notation just explained will be referred to as the matrix notation. In the next section the index notation will be introduced, and subsequently some presentations will be made in the form of two parallel columns, with the matrix notation used on the left and the index notation of the right. This will serve to illustrate the use of the index notation, and to emphasize the relative advantages of its use for the purposes at hand.

2. *Transformations. Tensors of order one.* Tensors are defined with respect to a certain type of transformations. In general tensor

analysis it is customary to deal with general functional transformations of the type $x^i = f^i(x', \cdots, x^n)$, where x^i is an abbreviation for the set of coordinates $x', \cdots, x^n$. It is sufficient to impose only the conditions of differentiability and reversibility. However, in this paper we limit ourselves to linear transformations of the variable coordinates.

Here we regard x^i as the homogeneous coordinates of a point in a flat space of $(n-1)$ dimensions. Most of the classical theory of tensors has as its basis an affine space in which non-homogeneous coordinates are used. Often the word "affinor" has been used for such a tensor. A tensor defined with respect to a general linear transformation of homogeneous coordinates may appropriately be called a "projective" tensor, but we shall speak of it simply as a tensor. The summation convention will be used throughout in this paper. This mexns that when the same index appears twice in a term this term stands for the sum of all the terms obtained by giving that index all the values it may take, this being n when we are working with a n-variable coordinate system.

Let the variable coordinates x^i be transformed into a new set $\bar{x}^i$ by the general non-singular linear transformation

$$(1) \qquad \overline{\underline{x}}^{\,n} = [t]_n^n \, \overline{\underline{\bar{x}}}^{\,n}, \quad \text{or} \quad \bar{x}^i = t_j^i x^j (i = 1, 2, \cdots, n).$$

For $n=3$ this transformation in detail is

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} t_1^1 & t_2^1 & t_3^1 \\ t_1^2 & t_2^2 & t_3^2 \\ t_1^3 & t_2^3 & t_3^3 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{bmatrix}, \quad \text{or} \quad \begin{aligned} x^1 &= t_1^1\bar{x}^1 + t_2^1\bar{x}^2 + t_3^1\bar{x}^3 \\ x^2 &= t_1^2\bar{x}^1 + t_2^2\bar{x}^2 + t_3^2\bar{x}^3 \\ x^3 &= t_1^3\bar{x}^1 + t_2^3\bar{x}^2 + t_3^3\bar{x}^3. \end{aligned}$$

The matrix $[t]_n^n$ is called the matrix of the transformation (1). An equivalent form of this transformation is obtained by taking the transpose of both sides of (1), getting $[x]_n = [\bar{x}]_n \overline{\underline{t}}^{\,n}_{\,n}$. Solving (1) for $\overline{\underline{\bar{x}}}^{\,n}$, we get

$$(2) \qquad \overline{\underline{\bar{x}}}^{\,n} = [T]_n^n \overline{\underline{x}}^{\,n}, \quad \text{or} \quad \bar{x}^i = T_j^i x^j,$$

where $[T]_n^n$ is the inverse of $[t]_n^n$; that is, T_j^i is the cofactor of t_i^j divided by $|t|$, the determinant of t_j^i, and $t_l^i T_k^l = T_l^i t_k^l = \delta_k^i$ ($\delta_k^i = 1$, if $i = k$; $\delta_k^i = 0$, if $i \neq k$). Under the transformation (1) the form $a_i x^i$ becomes $a_i t_k^i \bar{x}^k$. If the transformed form be thought of as $\bar{a}_i \bar{x}^i$, we have

$$(3) \qquad \overline{\underline{\bar{a}}}^{\,n} = \overline{\underline{t}}^{\,n}_{\,n} \; \overline{\underline{a}}^{\,n}, \quad \text{or} \quad \bar{a}_k = t_k^i a_i.$$

Solving (3) for $\overline{\underline{a}}^{\,n}$, we get

$$(4) \qquad \overline{\underline{a}}^{\,n} = \overline{\underline{T}}^{\,n}_{\,n} \; \overline{\underline{\bar{a}}}^{\,n}, \qquad a_k = T_k^i \bar{a}_i.$$

When the coordinates x^i are transformed by a linear transformation with matrix $[t]^n_n$, if there is a vector y^i which is transformed by the same matrix, then the vector y^i is called a *contravariant tensor of order one* with respect to the x^i coordinate system.

Law of contravariance for tensors of order one.

$$\text{(5)}\qquad \text{If}\quad \underline{\overline{x}}^{\,n}=[t]^n_n\,\underline{\overline{\bar{x}}}^{\,n}, \qquad \text{If}\quad \bar{x}^i=t_k^i x^k,$$

$$\text{then}\quad \underline{\overline{y}}^{\,n}=[t]^n_n\,\underline{\overline{\bar{y}}}^{\,n}. \qquad \text{then}\quad \bar{y}^i=t_k^i y^k.$$

In like manner we state the law of covariance for tensors of order one.

$$\text{(6)}\qquad \text{If}\quad \underline{\overline{x}}^{\,n}=[t]^n_n\,\underline{\overline{\bar{x}}}^{\,n}, \qquad \text{If}\quad x^i=t_k^i\bar{x}^k,$$

$$\text{then}\quad \underline{\overline{u}}^{\,n}=\underline{\overline{T}}^{\,n}_{\,n}\,\underline{\overline{\bar{u}}}^{\,n}. \qquad \text{then}\quad u_i=T_i^k\bar{u}_k.$$

Note that the transformation matrix of $\underline{\overline{u}}^{\,n}$ is the transpose of the inverse of $[t]^n_n$.

For brevity we say that a vector which transforms according to the law (5) is a *contravariant tensor of order one*, or a *contravariant vector*, it being understood that the vector is contravariant with respect to the x^i coordinate system under the transformation being considered. Similarly, a vector which transforms according to the law (6) is spoken of a *covariant tensor of order one*, or a *covariant vector*, We have thus two types of tensors of the first order; and while these two types are not distinguishable in the matrix notation, in the index notation they are distinguished by the position of the index; if the tensor of order one is contravariant a superscript is used as the index, and if it is covariant a subscript is used. Keeping in mind the above definitions, and recalling the terminology of matrix algebra, we may say: a^i is contravariant if a^i and x^i are cogredient, and a_i is covariant if a_i and x^i are contragredient. From the definitions (5) and (6) there follows the two theorems: If the components a^i and b^i of two contravariant vectors are identically equal in the x^i coordinate system, they are equal in all coordinate systems. The sum of two contravariant vectors is a contravariant vector. Similar statements may be made concerning two covariant vectors. Also we have the

Theorem. *The product a^ib_i of a contravariant vector and a covariant vector is invariant, that is, is the same in all coordinate systems.* For.

$$\text{(7)}\quad a^1b_1+\cdots+a^nb_n=[a]_n\,\underline{\overline{b}}^{\,n} \qquad\qquad a^ib_i=t_k^i\bar{a}^kT_i^j\bar{b}_j$$

$$=[\bar{a}]_n \underline{\overline{t}}^{n}_{n} \underline{\overline{T}}^{n}_{n} \underline{\overline{\bar{b}}}^{n} \qquad =\bar{a}^k t_k{}^i T_i{}^j \bar{b}_j$$

$$=[\bar{a}]_n [1]^n_n \underline{\overline{\bar{b}}}^{n} \qquad =\bar{a}^k \delta_k{}^j \bar{b}_j$$

$$=[\bar{a}]_n \underline{\overline{\bar{b}}}^{n} \quad . \qquad =\bar{a}^k \bar{b}_k.$$

It should be noted that the corresponding product of two contravariant vectors, or of two covariant vectors, is not in general invariant. A product like $a^i b_i$ is often referred to as a *scalar product*; another name for it is *tensor of order zero.* In the invariance of this type of product we have the germinal idea of the application of tensor algebra to the classical invariant theory.

3. *Tensors of order two.* Our next step is to examine what laws of transformation can be conveniently assigned to two-way matrices when the coordinates are subject to a linear transformation. A special case of a matrix is given by a column vector post-multiplied by a row vector; such a product may be anyone of three types: (1) the product of two covariant vectors; (2) the product of two contravariant vectors; or (3) the product of a contravariant vector and a covariant vector.

Case 1. Let a_i and b_i be two covariant vectors. Then by (6)

$$(8) \qquad [ab]^n_n = \underline{\overline{a}}^{n} [b]_n \qquad\qquad a_i b_i = T_i{}^k \bar{a}_k T_j{}^l \bar{b}_l$$

$$= \underline{\overline{T}}^{n}_{n} \underline{\overline{\bar{a}}}^{n} [\bar{b}]_n [T]^n_n \qquad\qquad = T_i{}^k T_j{}^l \bar{a}_k \bar{b}_l.$$

$$= \underline{\overline{T}}^{n}_{n} [\bar{a}\bar{b}]^n_n [T]^n_n.$$

Such a transformation of a matrix is referred to as a congruent equivalent transformation. In general, when the coordinates x^i are transformed by a linear transformation with matrix $[t]^n_n$, if there is a matrix $[a]^n_n$ which is transformed by the congruent equivalent transformation with matrices $\underline{\overline{T}}^{n}_{n}$ and $[T]^n_n$, then the matrix $[a]^n_n$ is a *covariant tensor of order two* with respect to the x^i coordinate system.

Law of covariance for tensors of order two.

$$(9) \qquad \text{If} \quad \underline{\overline{x}}^{n} = [t]^n_n \underline{\overline{\bar{x}}}^{n}, \qquad\qquad \text{If} \quad x^i = t_k{}^i \bar{x}^k,$$

$$\text{then} \quad [a]^n_n = \underline{\overline{T}}^{n}_{n} [\bar{a}]^n_n [T]^n_n. \qquad\qquad \text{then} \quad a_{ij} = T_i{}^k T_j{}^l \bar{a}_{kl}.$$

Case 2. Let a^i and b^i be two contravariant vectors. Proceeding analogously to Case 1, we see that $[t]^n_n$ plays the role of $\overline{\underline{T}}^{\,n}_{\,n}$.

Law of contravariance for tensors of order two.

(10) If $\overline{\underline{x}}^{\,n} = [t]^n_n \, \overline{\underline{\bar{x}}}^{\,n}$, If $x^i = t_k{}^i \bar{x}^k$,

then $[a]^n_n = [t]^n_n \, [\bar{a}]^n_n \, \overline{\underline{t}}^{\,n}_{\,n}$. then $a^{ij} = t_k{}^i t_l{}^j \bar{a}^{kl}$.

Case 3. Let a^i be a contravariant vector and b_j be a covariant vector. Then

(11) $[ab]^n_n = \overline{\underline{a}}^{\,n} \, [b]_n$ $a^i b_j = t_k{}^i \bar{a}^k T_j{}^l \bar{b}_l$

$= [t]^n_n \, \overline{\underline{\bar{a}}}^{\,n} \, [\bar{b}]_n \, [T]^n_n$ $= t_k{}^i T_j{}^l \bar{a}^k \bar{b}_l$.

$= [t]^n_n \, [\overline{ab}]^n_n \, [T]^n_n$.

A matrix transformation of this kind is called a similarity equivalent transformation. Nothing is gained by considering the product $a_i b^j$.

Law of transformation for mixed tensor of order two.

(12) If $\overline{\underline{x}}^{\,n} = [t]^n_n \, \overline{\underline{\bar{x}}}^{\,n}$, If $x^i = t_k{}^i \bar{x}^k$,

then $[a]^n_n = [t]^n_n \, [\bar{a}]^n_n \, [T]^n_n$. then $a_j{}^i = t_k{}^i T_j{}^l \bar{a}_l{}^k$.

The tensors $a_i b_j$, $a^i b^j$, and $a_i b^j$ are commonly referred to as the *outer products* of their respective factors. It should be noted that the outer product of two vectors is not in general commutative.

4. *Discussion of tensors and the index notation.* Tensors may be said to be matrices (which are arrays for which the operations of addition and multiplication are defined) which obey certain transformation laws when the variable coordinates are transformed in a given manner. Their importance lies in the fact that if a tensor equation is found to hold for one system of coordinates, this equation will hold when any transformation of the coordinates is made. In particular, if the components of a tensor are zero in one coordinate system, they are zero in all coordinate systems.

Because the index notation is commonly used in tensor algebra and analysis, it is often spoken of as the "tensor notation". However, the index notation does not have to be used in connection with tensors; it is used so only because workers in that field found it to be the most suitable symbolism for the purposes undertaken. Just because one uses the index notation one is not necessarily using tensors. In the

index notation the indices themselves are special symbols which may be interpreted as to play dual roles: (1) to distinguish between covariance and contravariance; and (2) to indicate the order in which the factors in a product are to be combined; the latter use of the index notation makes unnecessary some of the special symbols of the matrix notation, as that of the transpose of a matrix. Consider the bilinear form

$$[x]_3\,[a]_3^{\,3}\,\overline{\underline{y}}^{\,3}, \quad \text{or} \quad a_{ij}x^iy^j (i,j=1,2,3).$$

Note that in the representation on the right the order of the factors is immaterial. Also the index notation facilitates the study of higher forms, as the trilinear form $a_{ijk}x^ix^jx^k$, for which the corresponding matrix symbolism is rather complicated and only slightly developed. It will be found convenient to refer to n-way arrays, for which certain operations are defined, as n-way matrices, or briefly, as n-matrices; and to consider a tensor of the nth order as an n-matrix which transforms in accordance with a stated law. In this broad use of the word matrix, care should be taken in not assuming the existence of an extensive algebra for n-matrices as exists for 2-matrices.

5. *Relative or weighted tensors.* We now extend the term tensor to include entities like a^{ij}, where

(13) $$a^{ij}=|t|^{-w}t_k^{\ i}t_l^{\ j}\bar{a}^{kl} \quad \text{when} \quad x^i=t_k^{\ i}\bar{x}^k,$$

where $|t|$ is the determinant of $t_k^{\ i}$, and $|t|^{-w}$ means the determinant is raised to the $(-w)$ power. Equation (13) is equivalent to

$$\bar{a}^{kl}=|t|^{w}T_j^{\ k}t_j^{\ l}a^{ij}.$$

Our new concept includes the old as a special case when $w=0$. The set of quantities which satisfies (13) is called a *relative tensor*, and w is called its *weight*. The tensors we have previously considered are of weight zero, and are called *absolute tensors* when we wish to distinguish them from tensors of weight other than zero. Relative tensors of order zero are called *relative invariants*.

6. *Definitions and properties of tensors in general.* If there is a matrix $a^{ij\cdots k}$ $(i,j,\cdots,k=1,2,\cdots,n)$ such that

(14) $$a^{ij\cdots k}=|t|^{-w}t_p^{\ i}t_q^{\ j}\cdots t_r^{\ k}\bar{a}^{pq\cdots r} \quad \text{when} \quad x^i=t_k^{\ i}\bar{x}^k,$$

then $a^{ij\cdots k}$ is a *relative contravariant tensor*. If there is a matrix $a_{ij\cdots k}$ such that

(15) $$a_{ij\cdots k}=|t|^{-w}T_i^{\ p}T_j^{\ q}\cdots T_k^{\ r}a_{pq\cdots r} \quad \text{when} \quad x^i=t_k^{\ i}x^k,$$

then $a_{ij\cdots k}$ is a *relative covariant tensor*. If there is a matrix $a^{i\cdots j}_{k\cdots l}$ $\left({}^{i,\,\cdots,\,j}_{k,\,\cdots,\,l}=1,2,\cdots,n\right)$ such that

$$a^{i\cdots j}_{k\cdots l}=|t|^{-w}t_p{}^i\cdots t_q{}^jT_k{}^r\cdots T_l{}^s\bar{a}^{p\cdots q}_{r\cdots s}\quad\text{when}\quad x^i=t_k{}^ix^k, \tag{16}$$

then $a^{i\cdots j}_{k\cdots l}$ is a *relative mixed tensor*.* This includes (14) and (15) as special cases. The number of indices which appear in the symbol for a tensor indicates the *order* of the tensor. To hold the number of symbols to a minimu, $a^{i\cdots j}_{k\cdots l}$ will be used to represent either the components of the tensor or the tensor itself. The following is a list of the simpler operations that can be performed on tensors. The proof of these properties is based upon the definitions just given.

Addition. The addition of two relative tensors of the same order r and the same weight w, performed by adding corresponding components, produces a tensor of order r and weight w, which is called the sum of the two given tensors. This operation is commutative.

Multiplication. The multiplication of each component of a relative tensor of order R and weight W by every component of a relative tensor of order r and weight w produces a tensor of order $(R+r)$ and and weight $(W+w)$, which is called the product of the two given tensors. Equations (8) and (11) represent instances of tensor multiplication. Multiplication of tensors is associative, but in general is not commutative. The multiplication of each component of a relative tensor of weight w and order r by a scalar produces a relative tensor of the same weight and order. Such multiplication is commutative.

Contraction. This operation can be applied to any mixed tensor. It consists of setting a specified covariant index equal to a specified contravariant index; this gives a tensor of the same weight as the original tensor, but of order two less. To illustrate, from the mixed tensor of weight w and order two.

$$a_k{}^i=|t|^{-w}t_j{}^iT_k{}^l\bar{a}_l{}^j, \tag{17}$$

we obtain by setting $k=i$ the mixed tensor of order zero (a scalar invariant)

$$a_i{}^i=|t|^{-w}t_j{}^iT_i{}^la_l{}^j=|t|^{-w}\delta_j{}^la_l{}^j=|t|^{-w}a_l{}^l.$$

The scalar product a^ib_i (see equation (7)) may be considered as the contraction of the outer product a^ib_j.

*Alternatively stated, an array of scalars $a^{i\cdots j}_{k\cdots l}$ which obey the transformation law (16) is a tensor of the indicated order. These scalars may be what are commonly called points function of the variables x^i, or they may be constants.

Composition. The operations of tensor multiplication and contraction may be combined; for example, instead of thinking of the scalar product $a^i b_i$ as obtained from the outer product $a^i b_j$ by contraction, we may construct directly the former product with the two indices equal. Such combination of the operations of multiplication and contraction is called composition, and is used often in the construction of invariants.

REFERENCES

[1] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939.

[2] H. V. Craig, *Invariants and Elementary Mathematics*, NATIONAL MATHEMATICS MAGAZINE, Vol. XIII (1939), pp. 176-182.

On the Classification of Collineations in the Plane

By J. W. Lasley, Jr.
The University of North Carolina

1. *Introduction.* A ternary collineation is a projective correspondence between two two-dimensional forms in which like elements correspond. We shall consider here such a correspondence between two planes as point aggregates. To say that such a correspondence is projective is to say that it is one-to-one and of such a nature that to the points of a line in either plane there correspond points also on a line in the other plane. The lines associated in this way are said to correspond in the collineation. This corresponding of line to line provides us with the name collineation.

If the two planes in question coincide, we have what is called a collineation of a plane into itself. It is the purpose of this paper to classify[1] collineations of a plane into itself. This classification shall be made by means of invariants.

2. *Analytic representation.* If we refer the points and lines of a plane to a homogeneous projective reference frame, the relation bebetween the coordinates $x=(x_1, x_2, x_3)$ of a point P and those $x'=(x'_1, x'_2, x'_3)$ of the corresponding point P' is given[2] by

$$\rho x'_i = a_{ij}x_j \qquad (i, j=1,2,3), \tag{1}$$

where the a's are any numbers real or complex, where the determinant of the a's (written $|a_{ij}|$) is different from zero, and where the repeated index j indicates summation. The restriction $|a_{ij}|\neq 0$ enables us to solve (1) the other way around. This condition is imposed by the one-to-oneness of the correspondence. Such a collineation is known as non-singular.

The matrix of the a's (written $||a_{ij}||$) completely determines the collineation. The coordinates x of P are transformed by (1) into the coordinates x' of P'. We shall denote this by the expression $x'=a(x)$, where a symbolizes the matrix $||a_{ij}||$. The result of solving (1) the other way around may be denoted by $x=A'(x')$, where A' symbolizes the

[1] Baldus, R., "Zur Klassifikation der ebenenraumlichen Lollineationens", Sitzungberichte der Bayerischen Academie der Wissenschaften, December, 1928.

[2] Dœhlemann, K., Geometrische Transforationen, I Teil, Die Projektiven Transformationen nebst ihren Anwendungen, Sammlung Schubert XXVII, page 111.

matrix of cofactors transposed. These two relations connecting the points P with coordinates x with the corresponding points P' with coordinates x' constitute a point transformation of the plane into itself.

3. *The implied line transformation.* If we limit ourselves to those points x (i. e. points P with coordinates x) which lie on a line u and employ a harmonic reference frame, the corresponding points x' lie on a line u' for which $u' = A(u)$, where A symbolizes the matrix of cofactors. Solving the other way around we have the relation $u = a'(u')$, where a' symbolizes the original matrix transposed. These relations connecting lines u and the corresponding lines u' constitute an implied line transformation.

4. *Double elements.* Usually a point x is transformed into a point x' quite different from it. In order for the two points to be the same, it is necessary and sufficient that x satisfy a system of linear equations of matrix $a - \rho I$, where I is the identity matrix. Since such a system of equations always has a solution other than $x = (0,0,0)$, points of this sort always exist. They are known as double points of the collineation. Dually, a collineation always has double lines, which are lines u satisfying a system of linear homogeneous equations of matrix $a' - \sigma I$, where σ is the proportionality factor of the implied line transformation

$$(2) \qquad \sigma u_i = a_{ji} u'_j \qquad (i,j = 1,2,3,).$$

We thus see that the system of equations which double points x have to satisfy has the same determinant as that of the system of equations which the double lines u have to satisfy. In both cases the determinant must vanish, yielding the so-called characteristic equation of the collineation:

$$(3) \qquad |a - \rho I| \equiv \begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix} = 0,$$

an equation determining both ρ and σ; the former to be used in (1), the latter to be used in (2), so that when the primes are removed, the solutions x of (1) will give the double points and the solutions u of (2) will give the double lines of the collineation.

5. *Algebraic implication.* From the algebra of homogeneous linear equations[3] we may infer the following fundamental facts:

If the matrix $a - \rho I$ is rank 2, the elements in any row r_i (column c_i) are either all zeros or give the coordinates of a line (point) on a double

[3] Bocher, M., Introduction to Higher Algebra, page 47.

point (line). There are at least two lines (points) arising in this way determining a unique double point (line). The cofactors of some, although not necessarily a particular, row (column) give the coordinates of this double point (line). We have thus a unique double point (line) for each ρ for which the matrix $a-\rho I$ is rank 2.

If the matrix $a-\rho I$ is rank 1, the elements in any row r_i (column c_i) are either all zeros or give the coordinates of a line (point) of double points (lines). There is one, and just one, line of double points (point of double lines) arising in this way for each ρ for which the matrix $a-\rho I$ is frank 1.

If the matrix $a-\rho I$ is rank 0, the collineation is the identity. In this event every point (line) is a double point (line).

6. *Changes in the reference frame.* The collineation as defined is a geometric correspondence. For its study we have introduced an element quite foreign to it; namely, a coordinate system. Equations (1) give an analytic expression of the collineation. It does so with a definite choice of a reference frame. It is important to determine whether or not we can by some judicious choice of a reference frame materially simplify the analytic expression of the collineation. To this end we prove the following geometric

Theorem: *Every non-singular collineation has at least one double point and double line in united position.*

Proof: Since the system of homogeneous linear equations of matrix $a-\rho I$ has at least one non-trivial solution, there is always at least one double line in every non-singular collineation. Let us so choose our rerference frame that this line shall be made the third side s_3 of the reference triangle. A necessary and suffiicent condition for this is that $a_{31}=a_{32}=0$, $a_{33}\neq 0$. The collineation determines upon this double line a binary collineation.

$$\rho x_1' = a_{11}x_1 + a_{12}x_2$$
$$\rho x_2' = a_{21}x_1 + a_{22}x_2$$

which by argument similar to that above has always at least one double point, which will of necessity be on the double line. This establishes the theorem.

We next so choose our reference frame that this double point be made the first vertex V_1; retaining, of course, the incident line as s_3. A necessary and sufficient for this is $a_{21}=0$, $a_{11}\neq 0$, in addition to the conditions mentioned above.

We have thus reduced the matrix a to one in which all of the elements beneath the main diagonal are zeros. Although we have incor-

porated into the reference frame all of the double elements which we can be sure that the collineation possesses, we may make yet one further analytic simplification; namely, $a_{13}=0$. It may be that in securing the matrix a as outlined above a_{13} will come out to be zero. This will indicate that the third vertex $V_3=(0,0,1)$ is transformed into a point $V'_3=(0,a_{23},a_{33})$ on the line s_1, other than the point V_2. If $a_{13}\neq 0$, we may change our reference frame so as to secure $a_{13}=0$ in the new analytic expression of the collineation. In order to do this, we choose s_1 on V_3 and V_3'; keeping, of course, V_1 and s_3 at the union of double elements mentioned above. This is possible, unless (as may very well happen) V_3' is on s_2. A necessary and sufficient condition for this is $a_{23}=0$. In this event s_2 is a double line. The binary collineation determined upon s_2 is hyperbolic if and only if $a_{11}\neq a_{33}$. One of the double points of this binary collineation is V_1. If we choose the other as V_3, we shall secure $a_{13}=0$ in the resulting analytic expression of the collineation.

In case $a_{11}=a_{33}$ (and consequently the binary collineation on s_2 is parabolic), then s_3 is a double line on which the binary collineation is hyperbolic., unless a_{22} is also equal to a_{11}. This case is only notationally different from that just outlined, so by the mere expedient of interchanging the second and the third variables we may secure $a_{13}=0$ in the result. If in addition to the foregoing conditions we have also $a_{22}=a_{11}$, then the ternary collineation has a line of double points on a point of double lines. The reference frame may then readily be chosen so as to place V_1 at the point of double lines, s_3 at a double line and V_3 at a double point other than at V_1 on the line of double points. This makes both V_3 and V_3' on s_1, thus securing $a_{13}=0$ in the resulting analytic expression of the collineation. We have then the following analytic

Theorem: *Every non-singular collineation can be expressed analytically in the form*

$$\begin{aligned} \rho x_1' &= a_{11}x_1+a_{12}x_2 \\ \rho x_2' &= \qquad\quad a_{22}x_2+a_{23}x_3 \qquad\qquad a_{11}a_{22}a_{33}\neq 0, \\ \rho x_3' &= \qquad\qquad\qquad\quad a_{33}x_3 \end{aligned}$$

in which the matrix a *has zeros beneath the main diagonal and in the upper right hand corner.*

7. *Invariants.* Changes in the reference frame such as those outlined in the preceding section effect changes in the names of the points and lines of the collineation. They in no way affect the collineation itself, but only its analytic expression. Such changes in the analytic expres-

sion change the matrix a to a matrix b such that $b=pap^{-1}$, where p is the matrix of the substitution which changes the names of points x (or x') into y (or y'), and p^{-1} is its inverse. The matrix b is said to be the transform of a by p.

Let us ask: What features of the collineation, if any, are preserved under this transformation? Such features will then belong to the collineation rather than to the particular analytic expression used to describe it. These are called the invariants of the collineation under transformations of its analytic expression.

We observe first that the roots ρ of the characteristic equation (3) are invariants. In order to see this we have only to note[4] that if x is a double point of the collineation, then $x\rightarrow\rho x$ (read x is transformed into ρx) by a, $x\rightarrow y$ by p, $\rho x\rightarrow\rho y$ by p, $y\rightarrow x$ by p^{-1}, so that $y\rightarrow\rho y$ by $b=pap^{-1}$. Thus the presence of the same double point is revealed in the new analytic expression by means of the same ρ. This is what we mean by saying that ρ is an invariant under the transform of a by p. Since each root ρ of (3) is thus seen to be an invariant, it follows that the multiplicity of the roots of the characteristic equation is also an invariant.

Again, the rank of the characteristic matrix $a-\rho I$ is an invariant. This becomes apparent when we recall that the number of double points (lines) of a collineation and their distribution is a property of the collineation, and that neither can possibly be affected by the analytic expression of the collineation. Thus, if for a given value of ρ there arose a unique double point, this same value of ρ in some other analytic expression could not lead us to a line of double points.

We have then two invariants of a collineation under transforms of its analytic expression; namely, the multiplicity of the roots of its characteristic equation and the rank of any characteristic matrix.

8. *Classification.* On the basis of the two invariants above noted we now proceed to classify non-singular collineations of a plane into itself.

Case 1. *The characteristic equation has distinct roots.*

We shall suppose here and throughout the remainder of the discussion that our collineation is referred to the reference frame outlined in section 6, for which $a_{21}=a_{31}=a_{32}=a_{13}=0$. The roots of our characteristic equation are then a_{11}, a_{22}, a_{33}; all of which are distinct. Furthermore, we have seen that these roots, and consequently their differences $k_{ij}=a_{ii}-a_{jj}$ $(i,j=1,2,3)$, are invariant under all changes in the reference frame.

[4] Winger, R. M., An Introduction to Projective Geometry, page 311.

For $\rho = a_{11}$ the characteristic matrix $a - \rho I$ provides us with two lines $(0, k_{21}, a_{23})$ and s_3, since no k_{ij} is zero in this case. These lines meet in a unique double point, which is V_1. Dually, we have two points $(a_{12}, k_{21}, 0)$ and $(0, a_{23}, k_{31})$ on a unique double line not on V_1, which may be taken as s_1 in a new reference frame. This requires that $a_{12} = 0$ in the new analytic expression of the collineation.

For $\rho = a_{22}$, we have s_1 and s_3 as two lines determining a unique double point V_2. Dually, we have V_1 and point $(0, a_{23}, k_{32})$ determining a unique double line $(0, k_{23}, a_{23})$ not on V_2. This double line may be made s_2 thereby implying the condition $a_{23} = 0$.

Finally, for $\rho = a_{23}$, we have the lines s_1 and s_2 uniquely determining the double point V_3; and dually the points V_1 and V_2 uniquely determining the double line s_3.

The canonical form thus obtained is listed in the first line of the table below (see section 9) along with the configuration of double elements.

Case 2. a) *The characteristic equation has a double root a_{11} and a simple root a_{33}.*

b) *The rank of the characteristic matrix $a - a_{11}I$ is 2.*

For $\rho = a_{11}$, the characteristic matrix is rank 2, if and only if $a_{12} \neq 0$. In this event we have the two lines s_2 and s_3 determining the unique double point V_1. Dually, we have the two points V_1 and $(0, a_{23}, k_{31})$ determining the unique double line $(0, k_{13}, a_{23})$ not on V_2. This double line becomes s_2 upon requiring that $a_{23} = 0$.

For $\rho = a_{33}$, the characteristic matrix furnishes two lines $(k_{13}, a_{12}, 0)$ and s_2 on a unique double point V_3. Dually, we have two points V_1 and $(a_{12}, k_{13}, 0)$ determining a unique double line s_3.

Although we have incorporated into the reference frame all of the features outlined above, there remains yet another possible simplification in the analytic expression of the collineation. If we choose[5] the point $(a_{12}, 1, a_{12})$ as unit point, we thereby make $a_{12} = 1$ in the new expression of the collineation.

Again the results as to the canonical form and as to the double element configuration are listed in the table below—see line two.

Case 3. a) *The characteristic equation has a double root a_{11}, and a simple root a_{33}.*

b) *The rank of the characteristic matrix $a - a_{11}I$ is 1.*

The rank of the matrix $a - a_{11}I$ is rank 1, if and only if $a_{12} = 0$. There is a line of double points s_3; and dually a point of double lines $(0, a_{23}, k_{31})$ not on s_3, since the invariant $k_{31} \neq 0$. We now proceed to

[5] Woods, F. S., Higher Geometry, page 86.

make this point of double lines V_3. To do this we have merely to make $a_{23}=0$.

For $\rho = a_{33}$ the characteristic matrix is of rank 2, since $k_{13} \neq 0$. We have lines s_1 and s_2 determining a unique point V_3, the point of double lines obtained above. Dually we have the points V_1 and V_2 determining the unique line s_3, the line of double points above.

This collineation is known as an *homology*. The line of double points is called the *axis*. The point of double lines, not on it, is known as the *center*.

The canonical and the configuration of double elements are listed in line three of the table below.

Case 4. a) *The characteristic equation has a triple root* a_{11}.
b) *The rank of the characteristic matrix is 2.*

For $\rho = a_{11}$ we have $a_{12}a_{23} \neq 0$, since the characteristic matrix is rank 2. Lines s_2 and s_3 lead us to a unique double point V_1. Points V_1 and V_2 lead us to a unique double line s_3, on this double point.

Although there are no other double elements to incorporate into the reference frame, we may make yet a further simplification in the analytic expression. By choosing the point $(a_{12}a_{23}, a_{23}, 1)$ as unit point we may make a_{12} and a_{23} in the new system both equal to unity.

The canonical form and the configuration of double elements are both listed in line four of the table below.

Case 5. a) *The characteristic equation has a triple root* a_{11}.
b) *The rank of the characteristic matrix is 1.*

For $\rho = a_{11}$, we must have in order for the rank of the characteristic matrix to be 1 either $a_{12} \neq 0$, $a_{23}=0$ or $a_{12}=0$, $a_{23} \neq 0$. Since these two possibilities are only notationally different, we shall consider only the former. We have the line of double points s_2; and dually the point of double lines V_1 upon it.

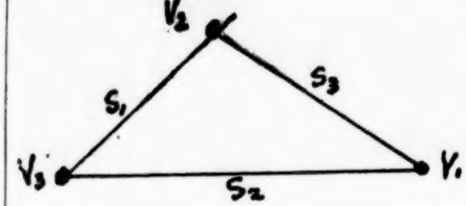
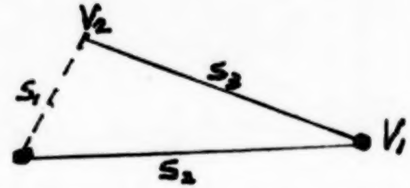
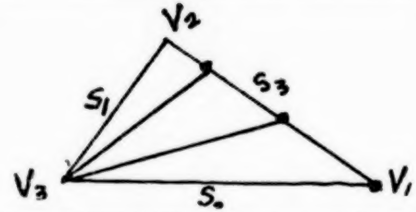
If now we choose the point $(a_{12}, 1, a_{12})$ as unit point, we reduce the coefficient a_{12} to unity in the new expression of the collineation.

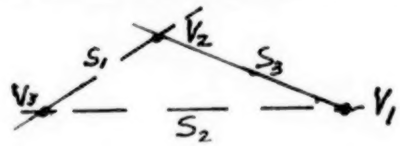
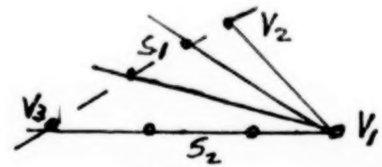
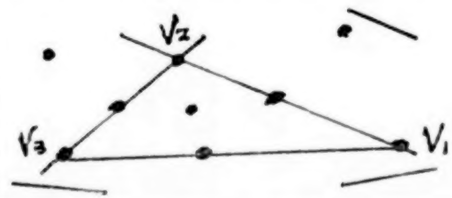
The canonical form and the configuration of double elements are listed in the fifth line of the table below. This collineation is known as an *elation*. The line of double points (axis) and point of double lines (center) are in united position.

Case 6. a) *The characteristic equation has a triple root* a_{11}.
b) *The rank of the characteristic matrix is 0.*

As stated in the last paragraph of section 5, the collineation is the identity.

The canonical form and the configuration of double elements is listed in line six of the table below. In the following table we list:

	Conditions	*Configuration of Double Elements*	*Jordan Canonical Form*
Case 1	Roots of the characteristic equation all distinct.		$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\alpha \neq \beta$, $\epsilon_{\alpha\beta} = 0$, $\beta \neq \gamma$, $\epsilon_{\beta\gamma} = 0$
Case 2	Characteristic equation has a double root and a simple root. Characteristic matrix for multiple root rank 2.		$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\alpha = \beta$, $\epsilon_{\alpha\beta} = 1$, $\beta \neq \gamma$, $\epsilon_{\beta\gamma} = 0$
Case 3 Homology	Characteristic equation has a double root and a simple root. Characteristic matrix for multiple root rank 1.		$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ $\alpha = \beta$, $\epsilon_{\alpha\beta} = 0$, $\beta \neq 0$, $\epsilon_{\beta\gamma} = 0$

	Conditions	Configuration of Double Elements	Jordan Canonical Form
CASE 4	Characteristic equation has a triple root. Characteristic matrix rank 2.		$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$ $\alpha=\beta$, $\epsilon_{\alpha\beta}=1$, $\beta=\gamma$, $\epsilon_{\beta\gamma}=1$
CASE 5 ELATION	Characteristic equation has a triple root. Characteristic matrix rank 1.		$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ $\alpha=\beta$, $\epsilon_{\alpha\beta}=1$, $\beta=\gamma$, $\epsilon_{\beta\gamma}=0$
CASE 6 IDENTITY	Characteristic equation has a triple root. Characteristic matrix rank 0.		$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ $\alpha=\beta$, $\epsilon_{\alpha\beta}=0$, $\beta=\gamma$, $\epsilon_{\beta\gamma}=0$

9. *Resume.* We now see not only that there is possible in all cases the canonical form obtained in section 6, in which the matrix of the collineation has zeros below the main diagonal and in the upper right hand corner, but we see also that in all cases the matrix may be reduced to the same simple Jordan[6] canonical form

$$a = \begin{pmatrix} \alpha & \epsilon_{\alpha\beta} & 0 \\ 0 & \beta & \epsilon_{\beta\gamma} \\ 0 & 0 & \gamma \end{pmatrix}, \tag{4}$$

where $\epsilon_{ij}=0$ for $i \neq j$ and $\epsilon_{ij}=0$ or 1 for $i=j$, and $\alpha\beta\gamma \neq 0$.

10. *Conclusion.* We have thus reduced non-singular collineations of a plane into itself to a common form, that of Jordan. The basis of this classification has been two absolute invariants under changes in the reference frame. The first of these is the multiplicity of the roots of the characteristic equation; the second is the rank of the characteristic matrix arising from this multiple root.

Considerations of incidence of double elements[7] have been limited to the proof of a single geometric theorem: Every non-singular collineation of a plane into itself has at least one double point and double line in united position.

In adjusting the reference frame to meet the special needs of the several cases characterized by the above mentioned invariants, a full use of duality has been made. The effect of these adjustments of the reference frame upon the analytic representation of the collineation has been constantly in the foreground. The simplified representation thus secured permits the geometric features of the collineation to emerge unhampered by needless algebraic complications. We are led not only to a unified expression of all possible cases in the canonical of Jordan, but also to a systematic means of distinguishing one case from another which make possible an easy remembrance of the Jordan canonical form of each.

[6] Bocher, *l. c.*, page 293.

[7] Browne, E. T., "On the Classification of Collineations in the Plane", American Mathematical Monthly, Vol. XL. (June-July, 1933) No. 6, page 333.

The Golden and Platinum Proportions

By E. T. Bell
California Institute of Technology

The Pythagoreans, and after them the Platonists, set great store by the socalled golden proportion

(1) $$6 : 9 :: 8 : 12,$$

in which 9 is the arithmetic mean, and 8 the harmonic mean between the extremes 6 and 12. It is said that Pythagoras brought this prolific curiosity from Babylon to Greece. A concise history of the numerology, including much of Pythagorean, Platonic, Neo-Pythagorean, Gnostic, Neo-Platonic, and mediæval science that issued from the golden proportion would fill a sizeable book.. Widely but not universally regarded as nonsense today, this aspect of classical mathematics and science is rather hastily passed over, like an embarrassing indecency, in the respectable histories of science and mathematics. Nor is it too conspicuous where it might reasonably be sought: in scholarly critiques of Greek philosophy. The resulting drama of Hellenic and Hellenized thought is not unlike—to use the hackneyed simile—*Hamlet* with Hamlet omitted, or perhaps more suggestively, like a one-ring circus without its clown.

It would do the great intelligences of the past a severe injustice to exhibit only a few of the scientific and metaphysical profundities they interred from the golden proportion. So we shall pass them all by for lack of space. To appreciate these masterpeices of the unaided reason as they merit, it is necessary to view several dozen of them together. Seen thus, it is evident that the cosmic sublimities extracted from the golden proportion by the Pythagoreans and their numerous successors are a compact body of closely reasoned doctrine, complete in itself and logically unimpeachable. The physics, the chemistry,

the cosmogony, and even the morphology of the Platonic "soul of the universe", constituting the several members of that harmoniously proportioned corpus, are unified by the supposedly incorruptible spirit of deductive reasoning. The logic by which they were deduced from explicitly stated postulates is as sound as any of Euclid's, indeed sounder than some of that immortal geometer's.

The moral is obvious and no doubt trite. Yet the art of inducing the good and the gullible to assimilate unlimited quantities of preposterous dogmas, simply because the patent absurdities of the dogmas in question have been deduced by irreproachable logic from nonsensical or unverifiable hypotheses, is by no means lost.[1] To any who may wish to indulge in this highly lucrative practise, the following threefold infinity of golden proportions is confidently recommended. The restriction to integer proportions is vital. Without integrality, a golden proportion has neither mathematical nor metaphysical interest, nor has it any possible pecuniary value.

The Pythagorean proportion (1) is the special case $x=6$, $y=12$ of

$$x : A(x,y) :: H(x,y) : y, \tag{2}$$

where $A(x,y)$, $H(x,y)$ are the arithmetic mean and the harmonic mean of the integers x,y. It is a simple exercise in diophantine analysis to show that the complete integer solution of (2) is

$$x=ab(2b-c), \quad y=abc, \tag{3}$$

where a,b,c are integers and b,c may be taken coprime. The corresponding arithmetic mean and harmonic mean are ab^2 and $ac(2b-c)$. If only positive integer solutions are desired, as by the ancient Pythagoreans and their modern disciples, $b>c/2$. There are thus a triple infinity of golden proportions in positive integers; (1) is the case $a=1$, $b=3$, $c=4$.

It will appear presently that the particular solution

$$x=137ab, \quad y=ab(2b-137) \tag{4}$$

of (2), in which a,b are arbitrary integers, may be of cosmic significance. The corresponding arithmetic mean is ab^2, as before. But the harmonic mean is now $137a(2b-137)$. The last might have been interpreted by an ancient Pythagorean as incontrovertible evidence that the intimate structure of the physical universe is in harmony with 137. In this, the ancients, as will be seen, have been followed by the modern Pythagoreans. If it be objected that 137 is a factor or maker of the harmonic mean only because it was insinuated into the generalized golden proportion (2) by taking $c=2b-137$, this is but another striking confirmation of the modern Pythagorean hypothesis that the mind elicits from nature only what the mind put into nature.

Squares and right triangles whose sides are measured by integers were of tremendous social, scientific, and philosophic importance to the ancient Pythagoreans and their successors. It has been suggested,[2] for example, that the mathematical eugenics in Plato's notorious Nuptial Number[3] is merely what a creative intelligence of the more metaphysical type might infer from the fact that, up to a trivial common multiplier, the unique integer solution of the simultaneous diophantine equations

$$x^2+y^2=z^2, \quad x^3+y^3+z^3=w^3$$

is $x=3$, $y=4$, $z=5$, $w=6$. The Pythagorean triangle 3,4,5 generates most of the metaphysics. The unicity of the solution is the point of mathematical interest.[4] It is not particularly easy to prove, and it is unlikely that any Greek mathematician or philosopher could have proved it.

With this antique reverence for integer squares in mind, it will be illuminating to quadrate the golden proportion (2), that is, to find all integers x,u,v,y such that

$$x^2 : A(x^2,y^2) :: H(x^2,y^2) : y^2, \tag{5}$$

$$A(x^2,y^2)=u^2, \quad H(x^2,y^2)=v^2.$$

As numerical examples that some observant Pythagorean might have noted,

$$425^2 : 625^2 :: 527^2 : 775^2,$$

$$391^2 : 289^2 :: 161^2 : 119^2,$$

in the first of which $(625^2,527^2)$ are the (arithmetic mean, harmonic mean) of $(425^2,775^2)$, and likewise for $(289^2,161^2)$ and $(391^2,119^2)$. The complete integer solution of (5) is

$$\begin{aligned} x &= tf(k^2+l^2)(k^2-2kl-l^2), \\ u &= tf(k^2+l^2)^2, \\ v &= tf(k^2-2kl-l^2)(k^2+2kl-l^2), \\ y &= tf(k^2+l^2)(k^2+2kl-l^2), \end{aligned} \tag{6}$$

where f,k,l are integers, f is arbitrary, and $t=1/16$ if k,l are both even, $t=1$ if one of k,l is even and the other odd, $t=1/4$ if k,l are both odd. This can be proved without difficulty.

Reminiscent of the mysterious (4), the choice $k=17$, $l=4$ yields a singularly provocative tetrad (6), for then both x and the harmonic v are multiples of the ubiquitous prime 137, which will be recognized as the value deduced by Eddington, "wholly from epistemological

considerations," for the fine-structure constant of spectroscopy. What this remarkable tetrad may signify for the higher numerology of modern physics is not yet entirely clear. But it is probably not too optimistic to anticipate an early enlightment: for no lesser a theoretical physicist than Eddington himself has declared his belief that "all the laws of nature that are usually classed as fundamental can be foreseen wholly from epistomological considerations." In a memorable goodby to gadgets,[5] he asserts that "an intelligence unacquainted with our universe, but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment. He would not deduce the particular events and objects of our experience, but he would deduce the generalizations we have based on them. For example, he would infer the existence and properties of sodium, but not the dimensions of the earth."

This courageous confession of faith raises an interesting problem in historical criticism. Some critical scholars have objected that "the golden proportion" is a misnomer, and insist that properly it is "the platinum proportion." Briefly, their argument runs somewhat as follows. By (4), the generalized golden proportion (2) also yields the cosmically potent 137 when suitably specialized. Moreover the two free integer parameters in (4) may be assigned values equal respectively to any two of the integer constants of the physical universe. In particular, the proton-electron ratio and the atomic number of platinum may be inserted by the mind into (2), to harmonize with 137 already implicit in the generalized golden proportion. Epistemologically, then, each of these three physical integers may be inferred from (2) without appeal to experiment, and therefore (2) is a necessary consequence of the world-mind structure. From this the critics infer that (2) is metallically, scientifically, and metaphysically richer than (5). Hence (2) is not epistemic gold but epistemic platinum.

But this alleged demonstration is fallacious in one respect and insufficient in another. The critics confuse the general (2) with the special (1)—not a damning oversight, perhaps, but still, damaging. Much graver is the failure of the critics to observe that the four squares in the quadrated golden proportion (5) render it transcendentally less comprehensible, and therefore infinitely more suitable, than (2) as the basis for a purely epistemological science. This almost completes the refutation. For, platinum being less easily obtainable than gold, it follows immediately from the preceding conclusion that (5), not (2), and certainly not the almost trivial (1), is the platinum proportion.

It has been objected to this that the ancients were unacquainted with platinum. But are we certain that they were unaware of the

existence of that costly element? By no means; in fact it is highly probable from "epistemological considerations" that they knew more about platinum than we are ever likely to discover. For it is undeniable that the more thoughtful intelligences of antiquity, though but slightly acquainted with "our universe," were perhaps only too familiar with "the system of thought by which the human mind interprets to itself the content of its sensory experience." Certainly the more famous of them were adept in that system of thought. It is therefore not improbable that they inferred "the existence and properties" of platinum. For surely platinum is no more difficult to epistemologize about than sodium; and either element is less elusive than the immaterial data of ethics reduced to confusion by the older numerologists with such conspicuous success.

The failure of the ancients to base their epistemological creations of the universe on the platinum proportion instead of on the golden as they did, must have cost purely reasoned science and abstract philosophy unimaginable treasures. When merely the classical theory of the constitution of matter,[6] for which the golden proportion was partly responsible, is recalled, it is poignantly evident that the loss to metaphysical science alone must be incalculable. Likewise for the lofty morals and elevating ethics accompanying some of the more inspiring syntheses of everything in space, time, and eternity. To take an elementary example, the apparition of the number 4 and its square in the universal constants t of the complete solution (6), of itself would have confirmed a devout Pythagorean in his childlike faith that Zeus was in Olympus and all was rational in the world. For 4, according to Pythagoras, is justice (according to Eddington it is the epistemologically necessary and sufficient number of dimensions of space-time); and squaring justice obviously returns the justice of the just to the just.

If the last seems fantastic[7] to us, we have but to contemplate the social arithmetic of the Nuptial Number [3] in its virginal purity, and to remember that this mystical application of rigorous mathematics to human affairs was offered in all seriousness to an insatiably credulous humanity by one of the most creative intelligences our believing and bemused race has been privileged to revere.

Having remembered this detail—only one of hundreds as discouraging from the somewhat depressing annals of the pure reason—we may begin to suspect that neither the history of science nor that of mathematics can be adequately or objectively presented wholly in terms of what are now accepted as successes. It would seem that not only the failures, some of which appear to have been first steps to success, must be recorded and analyzed if we are ever to understand the devious

mentality of our kind; the inspired nonsense and the crass absurdities also must be examined, and that with an impartial respect commensurate with the veneration once accorded them by human beings no more gullible than ourselves. Astrology, numerology, and all the other misadventures in ideas pursued almost wholly by the unaided intellect, no less than ad hoc epistemologies and puerile theologies, were as significant, as important, and as useful to our remoter ancestors in their search for what we now call scientific truth, as were phlogiston and the luminiferous æther to our nearer predecessors in the same endless quest.

Two thousand years hence, perhaps, phlogiston and æther will have joined the unmentionables in a bowdlerized history of science and reason, and "the glory that was Europe"—its science and mathematics of the three centuries following Gallieo and Newton—will have become an object of undiscriminating adulation to all congenital believers in the sanctity and the sanity of their past. In the meantime, ignoring anything we might not wish to inspect too closely in our own tradition back to Pythagoras, we may rest in the assurance that half a truth, unlike a half-truth, may sometimes be slightly better than none.

NOTES

1) See, for example, *Your Days are Numbered, a Manual of Numerology for Everybody.* By Florence Campbell, M.A. Fourth Edition, 1940. (Ray Long and Richard R. Smith, New York, Publishers.) In an appreciative Foreword the Publishers assure the prospective reader that by adding the numerical values of the letters in his name, followed by some "easy arithmetic," he will be enabled to select the right job, the right mate, etc. If the reader wants to know what to do or not do, the book, according to the Publishers, will tell him. The once-prosperous firm of Ray Long and Richard R. Smith dissolved and evaporated about 1932. Is it possible that one of the partners failed to vibrate to the other's number?. It would be interesting to know what educational institution conferred the M.A. degree on the talented authoress.

2) Grace Chisholm Young, *On the Solution of a Pair of Simultaneous Diophantine Equations Connected with the Nuptial Number of Plato.* London Mathematical Society, Proceedings, (2), 23, 1924, 27-44.

3) Plato's *Republic*, VIII, 546, B. C.

4) The integers 3,4,5,6 were especially prolific in Pythagorean and Platonic science.

5) A. S. Eddington, *Relativity Theory of Protons and Electrons*, Cambridge, 1936, p. 327.

6) For a brief account, see Plato's *Timæus.*

7) It may not. In an anthology of the more noteworthy things mathematicians of repute have said about mathematics in the past seventy years, which I hope some day to publish, there are many specimens even more remarkable than the hypothetical instance exhibited.

The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

An Experiment in Selecting Students According to Ability and Measuring Their Achievement by Common Examinations

By FLOYD S. HARPER
University of Nebraska

I. Introduction. The current emphasis on mathematics has brought about increased enrollments, especially in beginning courses, where the teacher is faced with the exceedingly complex task of instruction and evaluation of achievement of students of varying abilities and experience. The increased enrollments have necessitated corresponding increases in the teaching staff. The new staff members, no matter how much experience they may have, are faced with the problem of ascertaining the prevailing standards, which are not always too clearly defined. This situation may, with some justification, be likened to a factory attempting to turn out a product (the specifications of which are not clearly defined) from heterogenous materials, by workmen who have varying degrees of experience and training obtained at many different industrial plants.

It is not our purpose to discuss course objectives. They should, of course, be clearly defined (in the beginning courses especially) by the department in collaboration with the colleges or agencies whose students are being trained; however, the plan described does contribute to the analysis of this subject. We give a brief description of a method of classification and testing of a group large enough to necessitate at least three sections, which has made more effective teaching possible, and which has made the measurement of the achievement of the student more precise and meaningful. The methods described will be found applicable to larger groups requiring more sections.

II. Classification Test. Many experienced teachers believe that the inclusion in the same class of students of a high order of ability and training with those of low order of ability and training works to

the detriment of both groups. To stimulate the good students the teacher runs the danger of leaving the poor student bewildered and discouraged; and to adjust the standards of instruction to the ability of the poor student is certain to kill the interest of the good student.

Pre-study examinations,* on the basis of which the whole population of students in a given course may be sectioned according to their background and training in mathematics, seem to offer a partial solution to the problem of attaining a semblance of homogeneity within sections, especially in those institutions where enrollments are large enough to warrant such subdivision.

Such a pre-study examination is the Nebraska Mathematics Classification Examination, designed to measure the achievement of the students who matriculate at the University of Nebraska. On the basis of the scores received in this test the group was broken up into three Sections (I, II and III) in descending order of ability. Section III was made the smallest to permit of more individual attention. It may be of interest to report that Section II is the easiest to teach because of the greater homogeneity there. Section I, the most promising group, is the most stimulating to teach, but requires great effort to maintain and increase its interest.

III. Diagnostic Test. With the cooperation of the instructors† who have participated in repetitions of this experiment, a test covering the essential pre-requisites of the course is administered immediately after each class has been organized—usually during the second recitation period. This is a common examination given to the three classes which are always scheduled to meet at the same hour. The test determines the student's knowledge and proficiency in the fundamental operations and techniques upon which successful work in the course depends.

Uniformity in grading is achieved by a single instructor reading given questions for all of the classes. Each instructor makes a complete analysis of the scores for his section, student by student and question by question, to determine individual and class deficiencies. Remedial measures are then taken; later, short tests are given by each instructor to determine how rapidly and completely these deficiencies have been removed.

*Cox, H. M., *Pre-study Examinations in Mathematics*, NATIONAL MATHEMATICS MAGAZINE, Vol. XVII, No. 8 (May, 1943), pp. 351-359.

†Messrs. J. A. Eckstein, D. G. Glattly, F. S. Harper, J. A. Humann, C. L. Nelson and H. L. Schick, all of whom are experiemced teachers, some at the High School level and others at the College level. Various fields are represented: Business Organization, Coaching, Mathematics, Music, and General Science.

IV. Achievement Tests. Common examinations are of considerable value to instructors in obtaining a clearer picture of the accomplishments of a given section relative to the population as a whole; and the question arises as to whether this desirable practice can be followed where groups have been sectioned as to ability. We have done so in this experiment.

Presumably even the students in Section III have satisfied the minimum requirements and pre-requisites for the course in question, so most of them should, with proper application, cover the required material and attain the course objectives.

At approximately bi-weekly intervals, the instructor of Section III is asked to make out a set of questions which in his opinion cover the minimum essentials of the course to date. To this set, questions prepared by the instructor of Section II are added; it is intended that they will cover the same minimum essentials but will call for a greater ability to interpret and relate the facts and skills. Finally, the leader of Section I sees to it that the examination as a whole will challenge the best effort of his group. The examination is then reviewed carefully and worked through completely by each instructor so that ambiguities are removed and uniformity of notation and terminology is achieved.* A mimeographed copy of the examination is supplied to each student to insure that no difference will occur in transcribing by the various instructors and for the convenience of the student.

The students in each of the sections are urged to go as far as they can in the available time, and they are directed not to be disturbed by their inability to finish all of the questions. It is pointed out to them that the examination is designed to be a test in the real sense of the word for everyone. This is important as a challenge to the superior student, who deserves the right to show his superior ability and accomplishment, and as a spur to those in lower classes by showing them what can be done by the more able and better prepared students. I can conceive of nothing more likely to kill the interest of a good student than to be one of a group all of whom receive perfect scores on an examination which was not a real test of his ability.

The questions are weighted so as to make a total score in excess of 200 points, thus permitting finer grading of those questions which have sub-parts and those which have a sequence of steps leading to the final solution, each of which can be assigned some definite value.

*Incidentally, such examinations are bound to be superior to the results which could be achieved by any single instructor and they bring to a clear focus on repeated occasions the objectives of the course and the best methods of achieving them. Such common examinations and the scores obtained from them are of inestimable help to new instructors in determining the standards maintained by the department and in clarifying the objectives of the course.

The University of Nebraska

TABLE I

Mathematics Department

October 20, 1943.

Distribution of Scores—Class 14

Raw Scores	60 - 64	65 - 69	70 - 74	75 - 79	80 - 84	85 - 89	90 - 94	95 - 99	100 - 104	105 - 109	110 - 114	115 - 119	120 - 124
Section I												1	
Section II						1	1				1		
Section III	1		2	1	2	2	1		(2)	2	2	1	3
Totals	1		2	1	2	3	2		2	2	3	2	3
Grade	F								70 - 74				

Raw Scores	125 - 129	130 - 134	135 - 139	140 - 144	145 - 149	150 - 154	155 - 159	160 - 164	165 - 169	170 - 174	175 - 179	180 - 184	185 - 189
Section I	1	1			1	1		3	1	1		3	
Section II	4	2		2	(2)	2		1				4	
Section III													
Totals	5	3		2	3	3		4	1	1		7	
Grade	75 - 79							80 - 84					

Raw Scores	190 - 194	195 - 199	200 - 204	205 - 209	210 - 214	215 - 219	220 - 224	225 - 229	230 - 234	235 - 239	240 - 244	245 - 250
Section I	(1)	1	2		3	1	3		1		1	
Section II	2	1										
Section III												
Totals	3	2	2		3	1	3		1		1	
Grade	85 - 89				90 - 94				95 - 100			

Uniform grading of questions is strictly adhered to and one of the tabulation sheets illustrated below is filled out by sections showing the distribution of scores in each class and then totaled so show the complete distribution.

After studying the results obtained on the test and the course objectives, a minimum passing score can be fixed to correspond to the passing grade in the course. At this point we called for assistance from Mr. Cox, the director of the Bureau of Instructional Research at the University, who made a study of the ability pattern of the whole group as revealed by the classification test and other pertinent data. By means of this study a distribution of percentage grades was set up which, it is admitted, is somewhat arbitrary, but which is at least consistent with the only *a priori* measure of the mathematical ability of the group obtainable.

This pattern is not followed slavishly, but it does serve as a rough guide. Starting with the assumption that a given percentage of the group will receive grades of 95 and above, it is not difficult to locate the score corresponding to a grade of 95. In this manner the scores corresponding to 90, 85, etc. can be determined. Advantage is taken of such natural breaks as may occur in the distribution of scores in fixing these points so as not to disturb certain characteristic groupings. Then the final correspondence table* for all scores is set up (see Table I).

Where approximately uniform instruction is obtained, and with good tests, the median scores of the classes will reveal the gradations of ability of the separate sections.

V. Conclusions. The plan of sectioning students according to their ability is far from new, but common examinations with such sections are not generally used.

Our experience has demonstrated that common examinations are still possible and that they enable the instructor to see how his section fits into the total picture of the group, minimizing the possibility of an instructor obtaining a distorted opinion of the ability of a student who may show up favorably when compared with other students in his own class but whose abilities are more accurately evaluated when compared with those of the whole group.

*Each instructor is given a copy of the final analysis chart showing the achievement of his section relative to that of the other sections. He then makes a question analysis showing the number of his students who failed on each question so that characteristic weaknesses may be eliminated. An effort is also made to study the test itself so that poor or ineffective questions may be eliminated or modified before the test is administered to subsequent groups. Notes to this effect are filed away with the questions, in a manner similar to that described in *Better New Examinations from Old,* B. Clifford Hendricks and Otto M. Smith, Journal Chemical Education, Vol. 17, Number 12, December, 1940.

This point is of considerable significance and value to new instructors whose experience is limited as to time or locality, and to whom the problem of evaluation is replete with difficulties and uncertainties. Common examinations make possible mutually consistent grades from class to class, which is impossible where there is no common denominator. Moreover, the delicate task of determining whom to fail is made easier because it becomes more a matter of group judgment than an individual one.

NEW YORK, September 11, 1944.—Announcement has just been made of the formation of a new society for mathematical research, the Duodecimal Society of America, established as a voluntary, nonprofit organization incorporated in New York state. The purpose of the Society, as stated in its constitution, is "to conduct research and education of the public in mathematical science, with particular relation to the use of Base Twelve in numeration, mathematics, weights and measures, and other branches of pure and applied science."

"We count by tens for no better reason than that we happen to have ten fingers," says F. Emerson Andrews, president of the Society. "Philosophers and mathematicians have long agreed that twelve is a better and more efficient number base than ten; its actual use has been advocated at various times, notably by Herbert Spencer, the British philosopher, and Isaac Pitman, inventor of an early system of shorthand.

"In the past few years, interest has grown in this old idea. A group of experimenters have been trying it out in both practical and theoretical problems—from helping the U. S. Army transport service figure cargo cubages to working with factorials. For some kinds of problems, the savings introduced by dozen-counting are no less than astonishing. The growing number of experimenters and the importance of their findings made a common center for information and further research a necessity, and the Duodecimal Society of America was formed."

According to the literature of the society, counting by dozens can be learned by anyone in the space of about half an hour. Two new numerals are required—X to represent ten, and E for eleven. Decimals are replaced by duodecimals, so that *.4* means four-twelfths, and is a perfect third—a fraction for which there is no accurate decimal equivalent.

Officers of the Society are George S. Terry of Hingham, Mass., chairman of the board; F. Emerson Andrews of New York City, president; F. Howard Seely of Oakland, Cal., vice-president; and Ralph H. Beard of New York City, secretary and treasurer. The Society is located at 20 Carlton Place, Staten Island 4, New York.

Using Zero and One

By William R. Ransom
Tufts College

We hear too much (in the algebra class) about changing signs and about inverting.

In the problem $a-(b-c)$, the result $a-b+c$ *looks as if* the numbers b and $-c$ had "changed their signs". In the problem $a\div(b/c)$, the result $a\cdot(c/b)$ *looks as if* b/c had been "inverted". But something quite different really happened. If the student is led to think that algebra allows all sorts of changing about of symbols and looks-as-if rules, we need not be surprised if he begins to extemporize rules for changing symbols about in ways that soon hopelessly destroy the numerical values he should be working with.

Instead of changing signs and inverting, why not teach the beginner the legitimate devices of *adding zero* and *multiplying by one*, so that he can perform subtractions and divisions as simple *inverse* operations?

Thus to subtract $b-c$ from A, *prepare* the A by adding to it zero in the form $b-b+c-c$. From $A+b-b+c-c$ the b and the $-c$ can be subtracted by omission, leaving the $A-b+c$, without any indirection about changing signs or changing the operation from subtraction to addition. This way of subtraction goes back to fundamentals. There need be no rules for subtraction except this: From a sum of terms, any term that is present can be subtracted by omitting it; if the term to be subtracted is not visibly present, *prepare* for the subtraction by adding zero in the form term-to-be-subtracted plus its negative.

Similarly, in the case of division, the only rule needed is the analogous one: From a product of factors, any factor that is present can be divided out by omitting it; if the factor to be divided out is not visibly present, *prepare* for the division by first multiplying by one in the form factor-to-be-divided-out times its reciprocal.

To divide A by b/c, *prepare* the A by multiplying it by one in the form $b(1/b)c(1/c)$. From $A\cdot b(1/b)c(1/c)$, the factor b/c, which is the product $b(1/c)$ can be divided out by omission, leaving $A(1/b)c$ or Ac/b, without any indirection about inverting or changing the operation from division to multiplication.

Before undertaking to teach this method of division to a beginner, it is essential to make a simpler theory of fractions than the one usually taught.

The Egyptians confined themselves (we have been told) except for the fraction $^2/_3$, to those fractions whose numerators are one, such as $^1/_2$, $^1/_3$, etc., using $^1/_2+^1/_4$ instead of $^3/_4$, $^1/_2-^1/_{10}$ instead of $^2/_5$, etc. Let us call fractions with the numerator *one* (einheit) E-fractions. Our concept of $^1/_3$ is that it takes 3 of them to make 1. Generaliz8ng this concept, define an E-fraction as a number which being multiplied by its denominator gives one. (This immediately excludes $^1/_0$, and in the sequel excludes all division by zero.) Any other fraction is defined as the product of its numerator and its E-fraction. That is a/b is by definition $a(1/b)$, and by definition $b(1/b)=1$. With these two definitions is completed the whole theory of fractions and division. No further rules are required for either. The E-fraction $1/b$ is called the reciprocal of b. To get the reciprocal of $1/b$, take 1 in the form $b(1/b)$ and divide out the $1/b$, by omission, and we have b.

The rules for the multiplication of fractions become merely examples of the commutative and associative laws:

$$(a/b)(c/d)=a(1/b)c(1/d)=ac(1/b)(1/d)=ac(1/bd)$$

That $(1/b(1/d)=1/bd$ is shown by writing $1=1$ in the form

$$b(1/b)d(1/d)=bd(1/bd)$$

and dividing both sides, by omission, by bd.

The rules for the addition of fractions become merely applications of the distributive law: $a/b+c/d$ is *prepared* for addition by multiplying each by one in the form indicated, and we have

$$a/b+c/d=a(1/b)d(1/d)+c(1/d)b(1/b).$$

Here we have the two numbers ad and cb first multiplied by the common factor $(1/b)(1/d)$ and then added. According to the Distributive Law, the order of these operations can be interchanged: we may add first and then multiply. This reversal gives us $(ad+cb)(1/b)(1/d)$ which is written in the conventional form (of desired) as $(ad+cb)/bd$. It would be better if the conventional form could be abandoned. The "cancellatio asinorum", $(a+b)/ac=b/c$, would hardly occur to anyone who knew no rules for fractions except that $a(1/a)=1$.

Many teachers know the advantage of "clearing compound fractions" by multiplying by one. Thus to clear

$$\frac{(a/x)+(b/y)}{a/z}$$

multiply by 1 in the form xyz/xyz.

Adding zero is also useful in factoring. Instead of factoring x^2+6x+8 by trying to find pairs whose sum is 6 and whose product is 8, the device of adding zero in the form $9-9$ gives

$$x^2+6x+9-9+8=(x+3)^2-1=(x+3+1)(x+3-1).$$

This device *always* works, while the other will not work at all if the original 8 had been a 7.

Emphasis upon the fact that the only thing we can do to a quantity without destroying its value (apart from the changes of order allowed by the fundamental laws, commutative, associative, and distributive) is to add or subtract zero, or to multiply or divide by one, might do a good deal to check the wide spread destruction of good algebraic quantities by those reckless manipulators who change signs, invert, cancel, and otherwise mangle the literal expressions intrusted to them.

Note on the Teaching of Mathematical Induction*

By C. S. Carlson
St. Olaf College

It has been my experience in the teach ng of Mathematical Induction that practically every exercise in the ordinary text book gives a true formula and the pupils fail to see the urgent need of applying the proof. If a false formula is offered, the test for the cases of $n=1, 2, 3$, generally reveals the falseness of the formula and so the need is not met by applying the second part of the proof.

The following supplementary exercises have been set up and are true for $n=1, 2, 3, 4$, but fail for $n=5$. They are presented for whatever interest may be found in them, even though a bright college student should readily notice the difficulties presented by the denominator in all except one of them.

1. $$1+3+6+10+\cdots+\frac{n(n+1)}{2}=\frac{4(3n^2-n+1)}{13-n}$$

2. $$1^2+2^2+3^2+4^2+\cdots+n^2=\frac{(23n^2+100)(n-3)+252}{6}.$$

3. $$1+3/5+4/10+5/17+\cdots+1-n=\frac{4n(2n^2+109)}{127n^2+305n+12}$$

4. $$1+5+9+13+\cdots+(4n-3)=\frac{31n^2-155n+96}{(5-n)(2n-9)}$$

5. $$4+14+30+52+\cdots+n(3n+1)=\frac{2(29n^2-19n+12)}{12-n}$$

6. $$1/(1\cdot 4)+1/(4\cdot 7)+1/(7\cdot 10)+\cdots+\frac{1}{(3n-2)\,(3n+1)}$$
$$=\frac{50n-24-10n^2}{n(3n+1)(n-5)^2}.$$

*Presented at the meeting of the Minnesota section of Mathematical Association of America held at the College of St. Catherine's, May 15, 1937. Several requests for copies of these exercises have prompted offering them in print.

Brief Notes and Comments

Edited by
H. A. SIMMONS

8. *A Note on Newton's Theorem.* The collinearity of the midpoints of the diagonals of a complete quadrilateral can be based on the following theorem:

Theorem. If the triangles ABC and PQR are perspective, with P, Q, R on BC, CA, AB, and if XYZ and PQR are perspective, with X, Y, Z on QR, RP, PQ, then ABC and XYZ are perspective. (Johnson, *Modern Geometry*, p. 160.)

For, let the pairs of opposite points of the quadrilateral be DP, EQ, FR with DEF collinear, and let the diagonals EQ, FR meet in A, FR and DP in B, DP and EQ in C. Then ABC and PQR are perspective on the axis DEF. Take the midpoints of the sides of triangle PQR as X, Y, Z. Hence ABC and XYZ are also perspective, and the intersections of corresponding sides are collinear, but these are the midpoints of the diagonals of the complete quadrilateral.

Grinnell College. J. H. BUTCHART.

9. *On Interpreting a Formula.* It may be helpful to remind readers of this magazine of the interesting relations that exist between the physical and geometrical interpretations of certain formulas. The formula

$$\bar{x}_n=(x_1+x_2+x_3+\cdots+x_n)/n$$

with two others of the same type in y and z, may be looked upon as locating the center of gravity of n equal weights each of whose positions is determined by three coordinates having the same subscript. Or, the formula (and other similar ones) may be looked upon as locating a point having certain geometric relations to figures determined by *other* points, each of which is located by three coordinates having the same subscript.

When n is 2 the formula locates the midpoint of the line joining two points; when n is 3 it locates the intersection of the medians of

the triangle with its vertices at the three given points. When n is 4 the formula locates the intersection of two lines that join the midpoints of the opposite sides of a quadrilateral whether plane or skew; or it locates the midpoint of the line that joins the midpoints of the diagonals of the quadrilateral. Hence, these three lines all pass through one point and that point bisects each of them.

When n is 6 the formula may be written:

$$(1)\qquad n_6=\tfrac{1}{3}[(x_1+x_2)/2+(x_3+x_4)/2+(x_5+x_6)/2] = \tfrac{1}{3}(\bar{x}_2+\bar{x}_2'+\bar{x}_2'')=\bar{x}_3, \quad \text{or}$$

$$(2)\qquad =\tfrac{1}{3}[2(x_1+x_2+x_3+x_4)/4+(x_5+x_6)/2=(2\bar{x}_4+\bar{x}_2)/3, \quad \text{or}$$

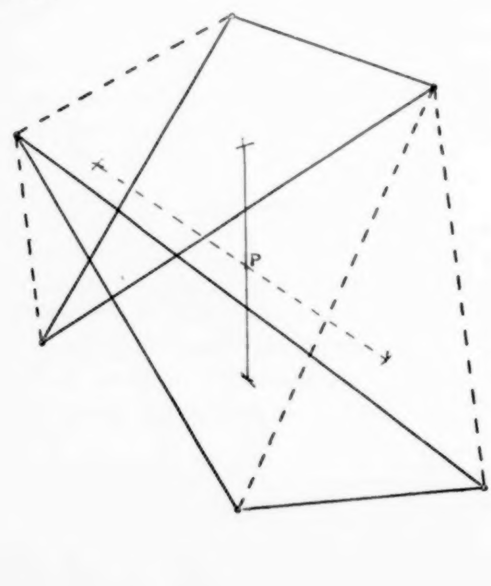
$$(3)\qquad =\tfrac{1}{2}[(x_1+x_2+x_3)/3+(x_4+x_5+x_6)/3]=\tfrac{1}{2}(\bar{x}_3+\bar{x}_3'=\bar{x}_2', \quad \text{or}$$

$$(4)\qquad =\tfrac{1}{6}[5(x_1+x_2+x_3+x_4+{}_5)/5+x_6]=(5\bar{x}_5+\bar{x}_6)/6, \quad \text{or}$$

$$(5)\qquad =\tfrac{1}{6}\left[\frac{x_1+nx_2}{n+1}+\frac{x_2+nx_3}{n+1}+\frac{x_3+nx_4}{n+1}+\frac{x_4+nx_5}{n+1}+\frac{x_5+nx_6}{n+1}+\frac{x_6+nx_1}{n+1}\right]=\bar{x}_6'.$$

Each of these ways of writing the formula leads to a theorem about six points. For example, from (3) arises the theorem:

If six points be used to determine two triangles with no coincident vertices the lines joining the intersections of the medians of all such pairs of triangles pass through a common point which bisects each of these lines.



The figure shows the geometry for this theorem for one distribution of the six points and two pairs of triangles. The line joining the intersections of the medians of another pair of triangles, without coincident vertices, determined by these points will, also, be bisected by the point P. Because of projective qualities this will be true of the figure in the plane of projection whether the points are all in one plane or not.

The form (5) of $\bar{n}_6$ shows that if the six lines joining six points in succession be all divided in the same ratio, internally or externally, the six new points thus determined will have the same center of gravity as the original six and any of the theorems corresponding to the forms (1) to (5) of $\bar{n}_6$ will determine the same point for either set of six points.

If the six points all lie in the same plane the theorems are concerned with the plane figures only. If five points lie in one plane and the other outside of the plane, besides the pentagonal pyramid the theorems will involve lines, triangles, quadrilaterals, pentagons, tetrahedrons and quadrangular pyramids. Thus, for every distribution of points there are particular corollaries to the original theorems.

As n increases both the number of theorems and the number of corollaries increase.

Lehigh University. JOSEPH B. REYNOLDS.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 505. Proposed by *Paul D. Thomas*, U. S. Navy.

Determine the surface generated by a variable circle having for diameter a diameter of a fixed hyperbola, the plane of the circle being perpendicular to the plane of the hyperbola.

Solution by *Earl V. Greer*, Bethany-Peniel College.

Let (x_1,y_1) be a point on the hyperbola $b^2x^2-a^2y^2=a^2b^2$, and let (x,y,z) be a point on the desired surface. Then (x,y,z) lies on the intersection of the sphere $x^2+y^2+z^2=x_1^2-y_1^2$ and the plane $xy_1=yx_1$. Eliminating x_1 and y_1 from these two equations, (remembering that $b^2x_1^2-a^2y_1^2=a^2b^2$) we derive the equation of the surface to be

$$(b^2x^2-a^2y^2)(x^2+y^2+z^2)=a^2b^2(x^2+y^2).$$

Also solved by the *Proposer*.

No. 528. Proposed by *N. A. Court*, University of Oklahoma.

The two sums of the squares of the distances of the vertices of two twin tetrahedrons from any point in space are equal.

Solution by *P. D. Thomas, RT 3c*, U. S. Navy.

If two of the parallel faces of a parallelepiped are $AD'CB'$; $A'DC'B$ and are such that AA', BB', CC', DD' are diagonals, then $ABCD$, $A'B'C'D'$ are the twin tetrahedrons inscribed in the solid. G is the common centroid of the tetrahedrons and of the parallelepiped. S and T are respectively the points of intersection of the diagonals in

the faces $AC'BD'$; $A'CB'D$, i. e. ST is a common bimedian of the twin tetrahedrons.

If P is any point in space we have by use of the median theorem

$$\overline{PA}^2+\overline{PB}^2=2\overline{PS}^2+\tfrac{1}{2}\overline{AB}^2,\quad \overline{PC}^2+\overline{PD}^2=2\overline{PT}^2+\tfrac{1}{2}\overline{CD}^2,$$

whence, adding,

$$(1)\qquad \overline{PA}^2+\overline{PB}^2+\overline{PC}^2+\overline{PD}^2=2(\overline{PS}^2+\overline{PT}^2)+\tfrac{1}{2}(\overline{AB}^2+\overline{CD}^2).$$

Similarly

$$\overline{PA'}^2+\overline{PB'}^2=2\overline{PT}^2+\tfrac{1}{2}\overline{A'B'}^2=2\overline{PT}^2+\tfrac{1}{2}\overline{AB}^2,\qquad (A'B'=AB)$$

$$\overline{PC'}^2+\overline{PD'}^2=2\overline{PS}^2+\tfrac{1}{2}\overline{C'D'}^2=2\overline{PS}^2+\tfrac{1}{2}\overline{CD}^2,\qquad (C'D'=CD)$$

and thus

$$(2)\qquad \overline{PA'}^2+\overline{PB'}^2+\overline{PC'}^2+\overline{PD'}^2=2(\overline{PT}^2+\overline{PS})^2+\tfrac{1}{2}(\overline{AB}^2+\overline{CD}^2).$$

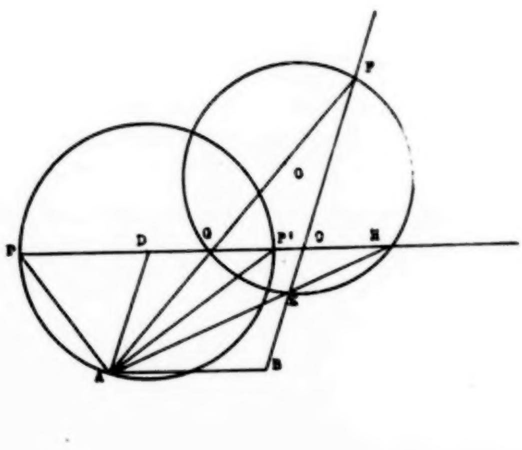
Since the right members of (1) and (2) are equal, the theorem is proved.

Also solved by the *Proposer*.

No. 525. Proposed by *V. Thébault*, Le Mans, France.

Two lines AEH, AFG drawn through the vertex A of a parallelogram $ABCD$ and symmetric with respect to the bisector of the angle A meet the sides BC and CD in the points E, F and G, H. Show that the points E, F, G, H lie on a circle, and determine the locus of the center of that circle when the angle between the two given lines varies.

Solution by *LeRoy Pietsch*, Student, Southern Methodist University.



We have angle

$$GFE = AFB = DAF = HAB = AHD = EHG.$$

Hence the segment EG subtends the same angle at the two points H and F. Therefore the points E, F, G, H, lie on a circle, say (O).*

Let P' be the trace of the bisector of the angle A on the side CD. We have angle

$$DAP' = P'AB = AP'D,$$

whence $DP' = DA$, and P' lies on the circle (D) having D for center and DA for radius. The external bisector of the angle GAH passes through the diametric opposite point P of P' on (D), whence P is the harmonic congugate of P' with respect to G, H; thus the points G, H are inverse with respect to (D), and therefore (O) is orthogonal to (D).

In a similar way it may be shown that (O) is orthogonal to the circle (B) having B for center and BA for radius.

Therefore the center O of (O) lies on the radical axis r of the two fixed circles (B), (D).

The common point A of (B) and (D) lies on r, hence the locus of O is the perpendicular from A to the line of centers BD, i. e., the diagonal BD of the parallelogram.

EDITORIAL NOTE. The locus of the center O may also be found as follows:

The parallel CUV through C to the diagonal BD is the harmonic conjugate of the diagonal CA with respect to the sides CB, CD. Hence CUV meets AEH, AFG in the harmonic conjugates U, V of A with respect to the pairs of points E, H and G, F. Therefore CUV is the polar of the point A for the circle (O).

As the circle (O) varies, the line CUV remains fixed. Hence the center O of (O) describes the perpendicular from A upon the line CUV, or upon the diagonal BD.—N. A. C.

The *Proposer's* solution is similar to the one by Pietsch.

Walter B. Clarke proved that the points E, F, G, H are concyclic.

No. 539. Proposed by *Paul D. Thomas*, U. S. Navy.

P and Q are the extremities of a pair of conjugate diameters of an ellipse, center O. Find (a) the locus of the centroid of the triangle OPQ, and (b) the locus of the foot of the perpendicular from O upon PQ.

*If AB is greater than CB and angle EAP' is sufficiently small, is is possible for H to fall between D and C and for E to fall on BC prolonged, contrary to their positions in the figure. But then angles GFE and EHG are supplementary, whence the conclusion is still valid.

Solution by *A. Sisk*, Maryville, Tenn.

Let the equation of the ellipse be given in the parametric form $x=a\cos\varphi,\ y=b\sin\varphi$. The eccentric angles of the points $P,\ Q$ differ by 90°, hence the coordinates of those two points are $P(a\cos\varphi,\ b\sin\varphi)$, $Q(a\sin\varphi,\ -b\cos\varphi)$, and the coordinates of the mid-point M of the chord PQ are $M[a(\sin\varphi+\cos\varphi)/2,\ b(\sin\varphi-\cos\varphi)/2]$.

If G is the centroid of the triangle OPQ, since $OG:GM=2$ we have for the coordinates $x,\ y$ of G

$$3x=a(\sin\varphi+\cos\varphi),$$

$$3y=b(\sin\varphi-\cos\varphi).$$

Eliminating φ between the two equations we obtain the required locus of the centroid to be the ellipse

$$9b^2x^2+9a^2y^2=2a^2b^2.$$

The equations of the line PQ and of the perpendicular OH from O upon PQ are, respectively,

$$b(\sin\varphi+\cos\varphi)x+a(\sin\varphi-\cos\varphi)y=ab,$$

$$a(\sin\varphi-\cos\varphi)x-b(\sin\varphi+\cos\varphi)y=0,$$

from which we obtain the two equivalent equations

$$\sin\varphi+\cos\varphi=ax/(x^2+y^2),$$

$$\sin\varphi-\cos\varphi=by/(x^2+y^2).$$

Eliminating φ we obtain the locus of the foot H of the perpendicular OH to be the quartic curve.

$$2(x^2+y^2)^2=(a^2x^2+b^2y^2).$$

Also solved by *Joseph S. Guérin, L. M. Kelly, H. J. Zimmerberg*, and the *Proposer*.

Guérin found the locus of G in the following way. Let R be the pole of the line PQ for the ellipse and N the point of the curve between O and R. The diagonal OR of the parallelogram $OPRQ$ passes through M and we have

$$ON^2=OM\cdot OR=OM\cdot 2OM=2OM^2,$$

hence $OM:ON=\sqrt{2}:2$. Now $OG:OM=2:3$, whence $OG:OM=\sqrt{2}:3$. Thus the point G corresponds to N in a homothecy having O for center and a given ratio. Hence the locus of G is an ellipse concentric with the given ellipse.

Kelly obtained the locus of G by projection.

Project the ellipse orthogonally into a circle. The conjugate diameters go into diameters at right angles. Hence $P'Q'$ will be of constant length. The centroid of OPQ will project into the centroid $O'P'Q'$. The projected centroid travels on a concentric circle, so the original must have as its locus an ellipse concentric with the given ellipse.

No. 544. Proposed by *Nev. R. Mind.*

If a light A is placed in the angle formed by two flat mirrors (V), (W), the plane (V) contains a point B, and only one, such that the ray AB after reflection in (V) and (W) returns to A. (See proposal 491, this magazine, October, 1943, p. 42.) Determine the path of B, when A moves on a fixed straight line.

Solution by ***Joseph S. Guérin***, Washington, D. C.

The point B is unique, for, A' and A'' being the symmetrics of A with respect to (V) and (W), the line $A'A''$ pierces the plane (V) in one and only one point, and this point is B. Let (L) be the fixed line described by A, and let (L') and (L'') be the reflections of (L) in (V) and (W) respectively. Now as A moves on (L), $A'A''$ slides so as always to intersect (L') and (L''), being also always parallel to a plane perpendicular to the line of intersection of (V) and (W). Consequently, the line $A'A''$ generates an hyperbolic paraboloid. It follows that the locus of B is the intersection of (V) with this hyperbolic paraboloid, and is, therefore a curve of the second degree other than an ellipse or circle.

Solved also by the *Proposer.*

No. 549. Proposed by *M. S. Robertson*, Rutgers University.

Let the distance from the center of the circle of radius r to the centroid of the area of a circular sector of angle 2θ be denoted by $f(\theta)$. In an elementary way show that $f(\theta)$ satisfies the functional equation

$$f(\theta)=f(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)$$

and obtain without calculus the solution

$$f(\theta)=\tfrac{2}{3}r\prod_{n=1}^{\infty}\cos(\theta/2^n)=\tfrac{2}{3}r\cdot\frac{\sin\theta}{\theta}.$$

Solution by ***J. S. Guérin***, Catholic University of America.

Let AOB be the given sector. The centroid G of the sector lies on OC the radius bisector. Similarly, the centroids G' and G'' of the

sectors AOC, BOC, lie on the bisectors of $\measuredangle AOC$ and $\measuredangle BOC$ respectively. But since G' and G'' must be symmetrical with respect to G and with respect to OC, $G'GG''$ is perpendicular to OC, and $OG=OG'\cos(\frac{1}{2}\Theta)$, or

(1) $$f(\Theta)=f(\tfrac{1}{2}\Theta)\cos(\tfrac{1}{2}\Theta).$$

In the same way, $$f(\tfrac{1}{2}\Theta)=f(\tfrac{1}{4}\Theta)\cos(\tfrac{1}{4}\Theta),$$

$$f(\Theta/2^2)=f(\Theta/2^3)\cos(\Theta/2^3),$$

$$\cdots\cdots\cdots$$

$$f(\Theta/2^{k-1})=f(\Theta/2^k)\cos(\Theta/2^k);$$

whence by substitution there results

(2) $$f(\Theta)=f(\Theta/2^k)\prod_{n=1}^{k}\cos(\Theta/2^n).$$

As k is made to approach infinity, $f(\Theta/2^k)$ approaches $2r/3$, because a sector of small angle is approximately an isosceles triangle; also the limit of the product in (2) is $\sin x/x$, by No. 508 (p. 132, December, 1943.) Hence, finally,

$$f(\Theta)=\tfrac{2}{3}r\prod_{n=1}^{\infty}\cos(\Theta/2^n)=\tfrac{2}{3}r\cdot\frac{\sin\Theta}{\Theta}.$$

Also solved by *Howard Eves*, and *H. J. Zimmerberg*.

No. 552. Proposed by *E. P. Starke*, Rutgers University.

If B is an end of the minor axis of an ellipse, determine P on the ellipse such that the chord BP has a maximum length. What is the value of the eccentricity when P is an end of a latus rectum?

Solution by *H. J. Zimmerberg*, University of Chicago.

Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$ and let B have coordinates $(0,-b)$. If P is the point (x,y) on the ellipse, we may seek, for convenience, to maximize the square of the distance BP, which is $x^2+(y+b)^2$. For a maximum we must have

(1) $$x\,dx/dy+(y+b)=0.$$

Assuming $x\neq 0$, it follows from the equation of the ellipse that $dx/dy=-ya^2/xb^2$, and substituting in (1), we find that the coordinates of P are

(2) $$x=\pm\frac{a^2}{a^2-b^2}\sqrt{a^2-2b^2},\quad y=\frac{b^3}{a^2-b^2}.$$

For P to lie on the ellipse and have its x-coordinate different from zero, we must have $y<b$, which implies that $a^2>2b^2$. For $a^2=2b^2$, P is the point $(0,b)$. Starting with the ellipse for which $a^2=2b^2$, and keeping b fixed, let a decrease. The new ellipse will lie entirely within the one for which $a^2=2b^2$. Thus, if $a^2\leqq 2b^2$, P is the point $(0,b)$; and for $a^2>2b^2$, P is given by (2).

If P is an end of a latus rectum, let c denote the distance from the origin to a focus. In this case the y-coordinate of P is equal to b^2/a, and using the relations $a^2=b^2+c^2$ and $e=c/a$ we have from the second relation in (2) that $e=c/a=b/c$. Hence, $c=ae$, $b=ce=ae^2$. Substituting these values in $a^2=b^2+c^2$, we have $e^4+e^2-1=0$. Solving by the quadratic formula we obtain

$$e=\sqrt{\frac{\sqrt{5}-1}{2}}\ .$$

J. S. Guérin (using polar coordinates) obtained the above results and showed that the maximum length of BP (when $e^2>\frac{1}{2}$) is a/e, which is the same as the distance from the center to the directrix. Also solved by *H. E. Fettis* (without the use of calculus), *Howard Eves*, *A. R. Thomas*, and *P. D. Thomas*.

EDITORIAL NOTE. When P is an end of the latus rectum, we had above the relations (a) $c/a=b/c$, and (b) $e^4+e^2-1=0$. (a) shows that c is the mean proportional between a and b. (b) may be put in the form $1 : e^2=e^2 : 1-e^2$, whence we see that e^2 satisfies the proportion designated as the "golden section." See notes and references of Problem No. 545 (this magazine, May, 1944, pp. 336-337.)

PROPOSALS

No. 556 (corrected). Proposed by *Howard D. Grossman*, New York City.

Prove that the limit, as $n\to\infty$, of the ratio

$$_{(n+1)(p+q)}C_{(n+1)p} : {}_{n(p+q)}C_{np}$$

is $(p+q)^{p+q}/p^pq^q$.

No. 575. Proposed by *V. Thébault*, Le Mans, France.

The four pairs of points A, A'; B, B'; C, C'; D, D' are marked on four parallel non-coplanar lines. The points P, Q, R, S divide the segments AA', BB', CC', DD' in the same ratio k; and the points P', Q', R', S' are the symmetrics of P, Q, R, S with respect to the mid-

points of those segments. Show that the sum of the volumes of the two tetrahedrons $PQRS$, $P'Q'R'S'$ remains constant when the value of k varies. Also determine the value of k for which the points P, Q, R, S, or the points P', Q', R', S' are coplanar.

No. 576. Porposed by *P. T. Thomas*, U. S. Navy.

$$I=\int \frac{a \cosh x+b \sinh x}{c \cosh x+d \sinh x}\, dx.$$

No. 577. Proposed by *Emory P. Starke*, Rutgers University.

Across one corner of a rectangular room two 4-foot screens are placed in such a manner as to enclose the maximum floor space. Determine their positions.

No. 578. Proposed by *Howard D. Grossman*, New York City.

What is the probability that somewhere in a deck of cards shuffled at random a 4-spot and a 7-spot (or any other two cards) of unspecified suits will come together?

No. 579. Proposed by *Walter B. Clarke*, San Jose, Calif.

The median of the triangle ABC issued from A meets the circumcircle again in D, and the tangents at A and D to that circle meet BC in E and F. Show that $AE=FD$.

No. 580. Proposed by *Howard Eves*, University of Syracuse.

The trilinear polars of the isotomic conjugates of any two points collinear with the centroid of the triangle are parallel.

No. 581. Proposed by *N. A. Court*, University of Oklahoma.

Given a coaxal pencil of spheres, the circle of intersection of a variable sphere of the pencil with the plane passing through a fixed line in the radical plane of the pencil and through the pole of the radical plane for the sphere considered, lies on a fixed sphere.

Bibliography and Reviews

Edited by
H. A. Simmons and P. K. Smith

Arithmetic for Adults. By Aaron Bakst. F. S. Crofts and Company, New York, 1944. vii+319 pages. $2.00.

Arithmetic for Adults with the subtitle *A Review of Elementary Mathematics* is suggestive of a shortcoming in our elementary educational system. The book is an attempt to remedy this. In his preface to the book, the author says "in consequence of faulty instruction many have never truly mastered the rudiments of arithmetic; perhaps few have sensed its possibilities as a tool in adult years". This statement gives the clue to the motive for writing the book. It is also suggestive of the methods of presentation that are to be employed.

The eighteen chapters of the book cover the subject from an explanation of our decimal system to instructions in the use of the slide rule.

The treatment of the four fundamental operations with numbers is unique in several respects. This author senses the need of understanding the meaning of the operations in order to be able to apply them to the solution of practical problems. As to the methods employed in bringing about this understanding there might be questions. For instance, the treatment of *fractional numbers* might be approached from a simpler standpoint than is done in this book. Most textbooks in arithmetic so far published have not done very well in making the transition from integral numbers to fractional numbers. Perhaps there is no one particular method that can take care of all of the difficulties in learning the meaning of the fundamental operations with fractional numbers and their applications to the solution of practical problems. The story of the development of number ideas and their expression shows that fractional number ideas were late in arriving. This was probably on account of inherent difficulties. This would seem to indicate that the approach should be as simple as possible.

The book might be improved by more diagrams to present to the eye the ideas of relative magnitudes of objects. Many learners are "eye-minded". Such diagrams and other picture forms are a great help.

On the whole, however, the book is a great improvement on most of the textbooks on arithmetic so far published. The main stress is laid on the meaning of number and its processes. It is comparatively easy to master the mechanical "skills" of the numerical processes, but to acquire ability to know what particular processes to use in solving practical problems requires a clear understanding of the nature of the processes. In this regard, the treatment in the book makes a strong appeal. The author is to be heartily commended for such a striking departure from the methods employed in most arithmetic textbooks.

Emeritus of Northern Illinois State Teachers College. S. F. Parson.

Learning to Navigate. By P. V. H. Weems and W. C. Eberle. Pittman Publishing Corporation, New York, 1943. viii+135 pages. $2.00.

The book gives a rather brief, non-mathematical discussion of the modern methods of navigation, including a chapter on air navigation. Omitted are discussions of Radio,

The Maneuvering Board, and Meteorology. Otherwise it takes up the usual subjects given in an up-to-date book on navigation, but does not go very deeply into any of the topics discussed, which may be an excellent idea as a first approach to this difficult subject. A good many interesting side-lights and historical notes are given, but on the whole it seems to be more of an outline or summary of the methods of navigation.

A good many of the explanations are easily followed and are adequate and complete. The diagrams accompanying them are clear in every detail. As an example of this are the diagrams comparing areas on the earth and on the Mercator chart at various latitudes. Quite often however, figures are given with little or no descriptive material to accompany them. For example, the diagram showing the principles upon which Mercator, Lambert, and stereographic charts are made. Also a rather complicated air navigation problem is proposed and the corresponding solution given by diagram, but no explanation of this appears, making it extremely difficult for the beginner to follow. This is also true of other vector diagrams appearing in the chapter on air navigation. In the figure showing the artist's representation of Principle of Celestial Navigation, the diameter of the circle of equal altitude should be twice the value given.

There are questions at the end of each chapter to bring out the material discussed in the chapter. But hardly ever are there any practical problems given, which could be worked out and plotted by the student, then checked with the answers supplied, and it is only by so doing that principles of piloting and celestial navigation are properly grasped. The excerpts from the air almanac and H. O. 214 are too brief to be of much use in working out the all important problems in line of position plotting, even if any such had been given, and this is perhaps the most serious objection of all. Tides and Current Tables are mentioned, but no excerpts of these appear so no problems are given. The student could hardly be expected to grasp this and other principles of navigation without having problems along these lines in the text, together with their answers, which he might solve and check.

This book would be excellent for the student who is in a hurry and wants to get a bird's-eye view of the subject. However, he would certainly need someone who knew the subject well to work along with him and to supplement a good many of the topics with further detail. Further, when it came to actually making up and using charts for work with problems in piloting, dead reckoning, and determination of the "fix", also for actual practice with the Dalton Mark VII Aircraft Plotter, he would certainly have to refer to other authorities for more complete information.

Tulane University. J. F. THOMSON.

Graphical Solutions. By Charles O. Mackey. John Wiley and Sons, New York. Second Edition, 1944. iv+152 pages.

The book under review provides graphical methods for the solution of the numerous computational problems which confront a scientific explorer in the practical application of his discipline. Although the book is designed primarily for the engineer, every scientist who uses numerical computation will find the work of considerable value.

The first edition, which appeared in 1936, has been enlarged by the introduction of new material. For example, the discussion of the construction and use of alignment charts has been expanded. Furthermore, "a complete explanation is given of the use of determinants in the graphical representation of equations of special form that contain as many as six variables. Projective transformation is discussed; this method may be used to construct alignment charts of convenient sizes and desirable proportions." In the first edition methods were presented by means of which curves are fitted to

experimental data which are non-periodic. In the present edition a new chapter has been added which extends the methods to periodic curves.

The book now consists of six chapters. The first discusses stationary adjacent scales, and the second sliding scales, which generalizes the theory of the ordinary slide rule, a device designed to multiply and divide numbers by sliding logarithmic scales. It is obvious that other scales will solve other problems and that the general device is one that is adaptable to many practical problems.

The third chapter considers network or intersection charts, which are graphical representations of equations of three or more variables. This leads naturally to the material in the fourth chapter which discusses the theory of alignment charts. The author defines an alignment chart as "the graphical representation of an equation in three variables, $f(u, v, w) = 0$, by means of three graphical scales (not necessarily straight), arranged in such manner that one straight line, called an index line, cut the scales in values of u, v, and w satisfying the equation."

The last two chapters discuss methods by means of which curves are fitted to empirical data. The important problem of determining the form of the curve is carefully discussed. Three methods are then described by means of which the parameters of the curve may be determined: (1) the method of *selected points;* (2) the method of *group averages* by *residual summation;* (3) the method of *least squares.* In the second of these chapters, as we have said above, the problem of fitting periodic curves to data is discussed. This chapter, considering the magnitude of the subject, seems very brief, comprising only eight pages. No attempt is made to show how the periods in the empirical data are to be discovered; nor are any of the simplifying devices inherent in special n-ordinate schemes discussed. The nature of the errors is also omitted.

The special value of the book will be found in the wealth of illustration which it contains. Many practical problems are fully discussed and the student is provided with an abundance of exercises upon which he may test his skill.

Northwestern University. H. T. DAVIS.

Marine and Air Navigation. By John Q. Stewart and Newton L. Pierce. Ginn and Company, New York. 472+12 pp. $4.50.

Among the large number of textbooks and manuals on navigation appearing currently this one by Stewart and Pierce stands in a class by itself.

Considering that it does not contain extensive tables it is the largest book on the subject, which this reviewer has ever seen. Every conceivable phase of air and sea navigation (which the authors correctly consider as essentially the same) is treated analytically and clearly.

A full page of acknowledgments appears in which no less than 79 persons are listed, to whom the authors express their thanks. The list includes the Secretary of the Navy, naval and air officers, authors, teachers and others having special knowledge of the subject.

Undoubtedly all of these people suggested changes, corrections, alternatives. Two comments belong here: First, that rigor and completeness are multiply assured; second, the wonder is that after such a barrage of criticism, the book is readable and interesting. It has literary quality with a distinct flavor of the wide, salty oceans and of the men "who go down to the sea in ships."

Many large, fine photographs are scattered throughout the book which give a "lift" to the reader. Illustrations and cuts by the hundreds illuminate the text.

It seems clear that the United States of America will for genrations be the leading air and sea power in the world. This means that many of the young men and women

in our universities will look to air and seafaring as a career. For them the authors have provided an excellent textbook for college courses in navigation. Moreover, the authors have had the genius to recognize that the subject has cultural interest to every educated person. Most people who travel by air or sea become interested in "track-following" in those trackless spaces. The tens of thousands of people who will have and fly their own planes after the war must know navigation.

Those who teach and those who learn, alike will find in this outstanding book the the rigorous thinking, the wealth of clean cut problems, the human and historical interest, the romance and philosophy of an old art and science which make an interesting and profitable course.

It is suggested that college teachers of Astronomy and Mathematics examine this book with the idea in mind of instituting full year courses in navigation in their departments.

Dearborn Observatory. OLIVER J. LEE.

Elementary Aviation. By L. E. Moore. D. C. Heath and Company, Boston, 1943. vii+222 pages. $1.60.

Here is a text on aerial navigation for high school students. It may be used for a one-semester course of ninety lessons, following two years of high school mathematics. The book contains a preface, a table of contents, eight chapters of text, and an appendix with four-place tables and six pages of *The American Nautical Almanac* for 1943 and five pages from *The American Air Almanac* for 1942. At the end is an index of about five pages.

The preface gives an exact statement as to the equipment needed by each student, suggests but does not insist upon a slide rule, and makes a surprising, and possibly unnecessary, number of helpful comments for teachers. The eight chapters have the following headings: I, *Introduction*; II, *Instruments;* III, *Meteorology;* IV, *Contact Flying;* V, *Dead Reckoning;* VI, *Radio Navigation;* VII, *Celestial Nevigation;* VIII, *General Review*. In every chapter the last lesson consists of review questions, with an oral quiz at the end of Chapter VI. Each lesson in the book starts with a few review problems on the preceding lesson. The last chapter has lessons of review questions on all that has gone before, followed by an examination (Part I, multiple choice questions; Part II, problems).

The author is a holder of C. A. A. Ground Instructor Certificate #53562 and a teacher of mathematics in a New York City high school. He knows his material, selects what is important from what is not, and makes it intensely interesting. Physics is introduced when needed. Newton's three laws of motion appear in the first four lessons. Vectors enter in the fourth, and are used successfully thereafter throughout the book. The author's diagrams are consistently good. Simple descriptions of several map projections are given, with statements as to the advantage and disadvantage of each. The actual planning of contact flights and the description of "flying the beam" are two of the most fascinating things in the book. The international Morse code appears, adapted for airway use by changing numeral symbols. There are numerous very practical problems, as there should be in a text of this type. A student will be well on his way to usefulness to his country as a flier when he finishes this book.

One must find some fault. The *compass rose* is mentioned in Lesson 12, but not defined until Lesson 15. There seems to be a mild disagreement between page 65 and Figure 33 as to whether United States highway No. 309 or 209 runs through Sellersville, Pa. And Pennsylvania should decide whether it has a town spelled East Petersberg (p. 86) or East Petersburg (Fig. 33). But these are minor points and it took considerable hunting to find them.

The cover of the book is effective, the printing clear and easy to read, the material well arranged. Even when the war is over, the fascination of this sort of text will remain. And now pardon us while we turn to Fig. 33 and plan a contact flight from New Brunswick, N. J. to Tamaqua, Pa.

Wellesley College. MARION E. STARK.

Intermediate Algebra. By Raymond W. Brink. D. Appleton-Century Company, Inc., New York and London, 1943. xii+268 pages. $1.50.

This book is written for students who have had one year of high school algebra. It would also serve well as a "refresher" for various types of students, including those whose morale would be improved by proceeding with the expressed hypothesis that they have merely forgotten what they thoroughly accomplished several years ago.

The text contains a good treatment from the point of view of drill of the four fundamental operations, factoring, fractions, linear equations in one or two unknowns with associated graphs, exponents, radicals, quadratic equations in one unknown and systems of them in two unknowns, ratio, proportion, variation, progressions, logarithms, and the binomial theorem. The book also contains a six page index and six tables with the following titles (excepting omissions of formulas):

Table I.	Squares, Cubes, Roots.	Table IV.	Present Value After *n* Periods.
Table II.	Common Logarithms.	Table V.	Amount of an Annuity.
Table III.	Compound Amount.	Table VI.	Present Value of an Annuity.

Tables III-VI are useful in connection with some of the material on progressions.

Presented with uniform clearness and with a sufficiency of well chosen illustrative examples, this book should please a teacher who wishes to give a drill course on the material in question.

Northwestern University. H. A. SIMMONS.

Basic Geometry. By George D. Birkhoff and Ralph Beatley. Scott, Foresman, and Company, Chicago, 1940.

Basic Geometry is a high school text, which includes most of the usual theorems, but which develops those theorems from an unusual set of fundamental principles. The five postulates concern line measure, intersection of two lines, equality of straight angles and similarity of triangles. Our present day knowledge of real numbers which was unavailable to Euclid is fundamental in the development of geometric theory as presented in this book. The appeal of the text to the student is first noticeable in the attractive format, and the pages of pictures one of which precedes each of the ten chapters. The style is informal at times almost conversational. Student's possible difficulties are recognized and discussed. Preceding the proof of a theorem, an analysis indicates the method of choosing pertinent assumptions and known theorems. Many theorems and ideas essential to the development of the theory are placed in the exercises, thus putting emphasis on original solutions and increasing the training in reasoning afforded by the course. Excellent exercises and discussions, which would be useful to a teacher of any text on high school geometry, illustrate the relationship between the use of logic in mathematical and non-mathematical situations. Abstract logic, non-Euclidean Geometry, inversion, equation of curves, symmetry, and other topics from more advanced mathematics are given brief mention. This book may seem too radical a departure from tradition to win immediate acceptance. Any teacher of mathematics can with profit inspect it, if only to be astonished that such a different text could be written.

Wright Junior College. RUTH M. BALLARD.